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Gravitational Motion and Radiation - I.

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Summary. — A method is established to solve Einstein's equations by successive approximations, uniqueness and compatibility at each step being secured by imposing de Donder's condition on the field. This turns out to restrict the freedom of choice of the sources in a way that determines their laws of motion, directly from the field quantities, without any use of integration around singularities. If the approximation parameter is the reciprocal velocity of light, the first non trivial approximation results in the constancy of the rest mass, and in Newton's law of attraction. The next step shows that there is no first-order correction to these laws. The third approximation then gives a small correction to the rest mass, which is found to be increased by the presence of other bodies. There is also a second-order correction to the acceleration (not explicitly evaluated here) which causes the perihelion advance in the two-body problem. The fourth approximation involves a contribution from the radiation field. It is found that in order to fulfil de Donder's condition, we must introduce into the field a term which is linear in the sources, although it does not appear in the so-called «linear approximation». This result may cast some doubts on the validity of the latter. In a further paper, we shall solve still higher approximations by means of an improved technique, and find that the fifth-order correction to the acceleration involves a non-conservative term. This last result may be taken as evidence for the reality of gravitational radiation.

1. – Introduction.

Exact solutions of Einstein's equations are known only in an exceedingly small number of cases. In most physical applications, the problem must be tackled by successive approximations, in which we assume that the solution is an analytical (1) function of some small parameter (*e.g.* the reciprocal velocity of light, or the gravitational constant) and thus can be expanded into a power series such as

$$g^{\mu\nu} = g_0^{\mu\nu} + g_1^{\mu\nu} + g_2^{\mu\nu} + \dots \quad (*)$$

where each term is by one order of magnitude smaller than the previous one.

It is then found that the equations to be solved are not, in general, compatible. Indeed, let us write Einstein's equations in the form

$$(1) \quad (g^{\mu\rho} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})_{,\alpha\beta} = 2\kappa g \mathfrak{D}^{\mu\nu} - Q^{\mu\nu} + g^{\mu\nu}_{,\alpha} g^{\alpha\beta}_{,\beta} - g^{\mu\alpha}_{,\alpha} g^{\nu\beta}_{,\beta} - g^{\mu\alpha}_{,\beta} g^{\nu\beta}_{,\alpha},$$

where

$$\begin{aligned} Q^{\mu\nu} = & g^{\alpha\beta}_{,\gamma} g^{\pi\eta}_{,\tau} [g^{\mu\eta} (-\frac{1}{2} \delta_\pi^\nu \delta_\alpha^\tau g_{\beta\eta} + \frac{1}{4} g^{\gamma\tau} g_{\alpha\pi} g_{\beta\eta} - \frac{1}{8} g^{\gamma\tau} g_{\alpha\beta} g_{\pi\eta}) - \delta_\alpha^\mu \delta_\pi^\nu \delta_\eta^\tau - \\ & - \delta_\alpha^\mu \delta_\pi^\eta g^{\gamma\tau} g_{\beta\eta} + \delta_\alpha^\mu \delta_\pi^\eta g^{\nu\tau} g_{\beta\eta} + \delta_\pi^\nu \delta_\alpha^\tau g^{\mu\eta} g_{\beta\eta} - \frac{1}{2} g^{\mu\eta} g^{\nu\tau} g_{\alpha\pi} g_{\beta\eta} + \frac{1}{4} g^{\mu\eta} g^{\nu\tau} g_{\alpha\beta} g_{\pi\eta}] \end{aligned}$$

and $g_{\alpha\beta} = g^{-1} g_{\alpha\beta}$.

When we compute $\frac{g^{\mu\nu}}{n}$, the right-hand side of (1) depends only $g_m^{\mu\nu}$, $m < n$, and thus can be considered as a known quantity. But the ten equations for $g^{\mu\nu}$ are not independent: indeed

$$[(g^{\mu\rho} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})_{,\alpha\beta}]_\nu \equiv 0$$

and therefore, equations (1) are compatible only if the four-divergence of their right-hand side is null. This in turn imposes four additional conditions on the previously found approximate solution, which can be fulfilled only for special values of $\mathfrak{D}^{\mu\nu}$. In such a way, compatibility of Einstein's field equations determines the laws of motion.

(1) B. BERTOTTI: *Nuovo Cimento*, **4**, 898 (1956); **6**, 755 (1957).

(*) Greek indices run from 0 to 3; Latin ones, from 1 to 3. It is convenient (2) to use as basic functions the contravariant density $g^{\mu\nu} = g g^{\mu\nu}$, where $g = \sqrt{-(\text{Det } g^{\mu\nu})}$. Pseudoquantities (*i.e.*, those of which the transformation laws involve some power of the Jacobian) are usually denoted by gothic letters.

(2) P. T. HOANG: *Compt. Rend.*, **246**, 61 (1958).

2. - General covariance.

When the laws of motion are satisfied, we remain with only six independent equations for the ten $\underset{n}{g^{\mu\nu}}$: this corresponds to the possibility of giving to four of them arbitrary values, by an infinitesimal co-ordinate transformation of the n -th order. In order to secure uniqueness of the solution and to get rid of apparent fields, we must impose on $\underset{n}{g^{\mu\nu}}$ four additional conditions. These can be made fully covariant by introducing, side by side with $\underset{n}{g^{\mu\nu}}$, an auxiliary flat (but not necessarily Minkowskian) metric $\gamma^{\mu\nu}$ ⁽³⁾ which we shall identify with the « zeroth » approximation $\underset{0}{g^{\mu\nu}}$. It is then convenient to choose, as additional conditions, de Donder's relations

$$\underset{n}{g^{\mu\nu}}_{,\nu} = 0 ,$$

where the comma now denotes a covariant derivative with respect to the affinity belonging to $\gamma^{\mu\nu}$. It is easily shown, by generalizing a theorem of FOCK ⁽⁴⁾, that de Donder's relation prevents any *infinitesimal* co-ordinate transformation, modifying only $\underset{n}{g^{\mu\nu}}$, ($n > 0$), besides those generated by harmonic functions ⁽⁵⁾, and thus, under suitable conditions at infinity ⁽⁶⁾, secures the uniqueness of the solution, once $\gamma^{\mu\nu}$ has been chosen ^(*).

We shall assume, in this paper, that the velocities are small quantities of the first order. (This is equivalent to the choice of c^{-1} as the expansion parameter. However, it is more convenient to take $c = G = 1$). Accelerations, and masses are therefore of the second order. It is easily seen that the only explicit dependence of the field quantities on time is through the positions and velocities of the sources. It follows that a time derivative of a field quantity is, in general, by one order of magnitude smaller than a space derivative. This, however, is due only to the choice of the expansion parameter (which is Lorentz invariant) and *does not impair at all the general covariance of the theory* as stated by some authors ^(1,8).

⁽³⁾ N. ROSEN: *Phys. Rev.*, **57**, 147 (1940).

⁽⁴⁾ V. A. FOCK: *Theory of space, time and gravitation* (in Russian, Moscow, 1955), p. 441.

⁽⁵⁾ L. LANDAU and E. LIFSHITZ: *The Classical Theory of Fields* (English translation, Cambridge, Mass., 1951), p. 324.

⁽⁶⁾ A. TRAUTMAN: *Bull. Acad. Pol. Sc.*, **6**, 407 (1958).

^(*) Without the introduction of $\gamma^{\mu\nu}$, de Donder's condition is not covariant, thus seriously impairing the generality of the theory ⁽⁷⁾.

⁽⁷⁾ L. INFELD: *Rev. Mod. Phys.*, **29**, 398 (1957).

⁽⁸⁾ P. HAVAS: *Phys. Rev.*, **108**, 1351 (1957).

3. – Approximation procedure.

We shall assume that matter is present in the form of N particles of rest mass $\overset{A}{M}_0$ and co-ordinates $\overset{A}{\xi}^k = \overset{A}{\xi}^k(t)$, $A = 1, \dots, N$. We have (8-10):

$$(2) \quad \mathfrak{T}^{\mu\nu} = \sum_{A=1}^{A=N} \overset{A}{M}_0 \frac{d\xi^\mu}{d\sigma} \frac{d\xi^\nu}{d\sigma} \frac{d\overset{A}{\sigma}}{dt} \delta(\mathbf{x} - \overset{A}{\xi}),$$

where

$$d\overset{A}{\sigma} = \gamma_{\mu\nu} d\xi^\mu d\xi^\nu.$$

The Lorentz-invariance of the rest mass follows from the fact that $\mathfrak{T}^{\mu\nu} dt dV$ is a genuine tensor.

It is easily seen that the first non-trivial terms of $g^{\mu\nu}$ are $\overset{0}{g}^{00}$, $\overset{0}{g}^{0k}$ and $\overset{0}{g}^{kl}$. We shall therefore call n -th approximation the computation of $\overset{n+1}{g}^{00}$, $\overset{n+2}{g}^{0k}$ and $\overset{n+3}{g}^{kl}$.

It will turn out that

$$(3) \quad \overset{n+1}{g}^{00,0} + \overset{n+2}{g}^{0k,k} = 0,$$

will give the $(n-1)$ -th order corrections to the masses, and

$$(4) \quad \overset{n+2}{g}^{k0,0} + \overset{n+3}{g}^{kl,l} = 0,$$

the $(n-1)$ -th order corrections to the accelerations.

As we are dealing with a problem having no special symmetry, it is convenient to chose $\gamma^{\mu\nu} = \eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowskian metric: $\eta^{00} = 1$, $\eta^{0k} = 0$, $\eta^{kl} = -\delta^{kl}$. With the help of de Donder's conditions, Einstein's equations now read

$$(5) \quad \nabla^2 g^{\mu\nu} - \ddot{g}^{\mu\nu} = -16\pi g \mathfrak{T}^{\mu\nu} + \Theta^{\mu\nu},$$

where

$$\Theta^{\mu\nu} = Q^{\mu\nu} + (g^{\alpha\beta} - \eta^{\alpha\beta}) g^{\mu\nu}_{,\alpha\beta}.$$

The ten equations (5) are independent, and therefore always compatible. If we neglect $\Theta^{\mu\nu}$, we get the so-called « linear approximation » which is the basis of usual computations on gravitational radiation (11). It is therefore

(9) J. LUBAŃSKI: *Acta Phys. Pol.*, **6**, 356 (1937).

(10) W. TULCZYEW: *Bull. Acad. Pol. Sc., Cl. III*, **5**, 279 (1957).

(11) L. LANDAU and E. LIFSHITZ: loc. cit., p. 330.

interesting to examine whether Einstein's equations (5) have exact solutions, in which the part of $g^{\mu\nu}$ which is linear in the masses is identical with the « linear approximation ».

We shall therefore write

$$g^{\mu\nu} = h^{\mu\nu} + \tilde{s}^{\mu\nu},$$

where

$$\nabla^2 h^{\mu\nu} - \ddot{h}^{\mu\nu} = -16\pi g \mathfrak{D}^{\mu\nu}$$

is the linear approximation, and

$$\nabla^2 \tilde{s}^{\mu\nu} = \ddot{\tilde{s}}^{\mu\nu} + \Theta^{\mu\nu}.$$

The equation for $h^{\mu\nu}$ can be solved by the use of Lienard-Wiechert potentials:

$$h^{\mu\nu} = 4 \sum \frac{M_0 g}{|R_\lambda(d\xi^\lambda/d\sigma)|} \frac{d\xi^\mu}{d\sigma} \frac{d\xi^\nu}{d\sigma}; \quad R^\lambda R_\lambda = 0,$$

where $R^\lambda = x^\lambda - \xi^\lambda$, and we have omitted the superscripts A .

Defining the velocity

$$v^k = \frac{d\xi^k}{dt}; \quad v^0 = 1,$$

and the « effective source »

$$M = M_0 \frac{dt}{d\sigma} g,$$

this can be written as

$$h^{\mu\nu} = 4 \sum \frac{M v^\mu v^\nu}{R(1 \pm \dot{R})} \quad \text{computed at time } (t \mp R),$$

where $R = \sqrt{R^k R^k}$ and the \pm sign is to be taken according to the choice of retarded or advanced potential. For more generality, we shall take a linear combination of these potentials, such as

$$h^{\mu\nu} = \frac{1}{2} (h_{\text{ret}}^{\mu\nu} + h_{\text{adv}}^{\mu\nu}) - \frac{\varepsilon}{2} (h_{\text{ret}}^{\mu\nu} - h_{\text{adv}}^{\mu\nu}),$$

where ε is an arbitrary constant. It may be shown (12-14) that

$$\frac{1}{2} (h_{\text{ret}}^{\mu\nu} + h_{\text{adv}}^{\mu\nu}) = 4 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{d^{2n}}{dt^{2n}} \sum M v^\mu v^\nu R^{2n-1},$$

(12) A. S. EDDINGTON: *The Mathematical Theory of Relativity* (Cambridge, 1924), p. 249.

(13) F. R. CROWNFIELD jr. and P. HAVAS: *Phys. Rev.*, **94**, 471 (1954).

(14) B. BERTOTTI: *Nuovo Cimento*, **2**, 231 (1955).

and

$$-\frac{1}{2} (\mathfrak{h}_{\text{ret}}^{\mu\nu} - \mathfrak{h}_{\text{adv}}^{\mu\nu}) = 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dt^{2n+1}} \sum M v^\mu v^\nu R^{2n},$$

where all quantities are now to be computed at the field time t .

In these expressions, we suppose that the co-ordinates ξ^k and velocities v^k are known (initial conditions), and that the masses and accelerations have to be expanded

$$\begin{aligned} M &= m + \underset{1}{m} + \underset{2}{m} + \dots, \\ \dot{v}^k &= a^k + \underset{1}{a}^k + \underset{2}{a}^k + \dots \quad (*) . \end{aligned}$$

4. – First approximation.

We have

$$\underset{2}{g}^{00} = \underset{2}{h}^{00} = 4 \sum \frac{m}{R},$$

$$\underset{3}{g}^{0k} = \underset{3}{h}^{0k} = 4 \sum \frac{mv^k}{R}.$$

Application of (3) gives

$$\sum \frac{\dot{m}}{R} = 0,$$

which can be satisfied if and only if $\dot{m} = 0$. The rest mass is therefore a constant at this approximation.

Further, we have

$$\underset{4}{h}^{kl} = 4 \sum \frac{mv^k v^l}{R},$$

but in order to get $\underset{4}{s}^{kl}$, we first have to compute $\underset{4}{\theta}^{\mu\nu}$:

$$\left\{ \begin{array}{l} \underset{4}{\theta}^{00} = \frac{7}{8} \underset{2}{g}^{00,n} \underset{2}{g}^{00,n}, \\ \underset{4}{\theta}^{0k} = 0, \\ \underset{4}{\theta}^{kl} = \frac{1}{8} \delta^{kl} \underset{2}{g}^{00,n} \underset{2}{g}^{00,n} - \frac{1}{4} \underset{2}{g}^{00,k} \underset{2}{g}^{00,l}. \end{array} \right.$$

(*) Note that $\underset{n}{a}^k$ and $\underset{n}{m}$ are small quantities of order $(n+2)$.

From this, we deduce (15):

$$\sum_4 S^{kl} = \frac{1}{2} \sum m^2 \left[\frac{\delta^{kl}}{R^2} - \frac{\partial}{\partial \xi^k} \frac{\partial}{\partial \xi^l} (\log R) \right] + \sum'_{A,B} \overset{A}{m} \overset{B}{m} (2\delta^{kl} S^{nn} - 4S^{kl}),$$

where terms with $A = B$ are excluded from \sum' and

$$S^{kl} = \frac{\partial}{\partial \overset{A}{\xi}^k} \frac{\partial}{\partial \overset{B}{\xi}^l} [\log (\overset{A}{R} + \overset{B}{R} + D)],$$

D being the distance between particles A and B .

Application of (4) gives

$$\sum_A \frac{\overset{A}{m} \overset{A}{a}^k}{R} - \sum'_{A,B} \overset{A}{m} \overset{B}{m} \left(S^{kl,l} - \frac{1}{2} S^{ll,k} \right) = 0.$$

We tentatively try a solution such as

$$\overset{A}{a}^k = \sum_B \overset{B}{m} \overset{AB}{F}^k,$$

whence

$$\sum'_{A,B} \overset{A}{m} \overset{B}{m} \left(\frac{\overset{AB}{F}^k}{R} - S^{kl,l} + \frac{1}{2} S^{ll,k} \right) = 0.$$

The coefficient of $\overset{A}{m} \overset{B}{m}$ is

$$\frac{\overset{AB}{F}^k}{R} + \frac{\overset{BA}{F}^k}{R} - S^{kl,l} - S^{lk,l} + S^{ll,k} = \frac{\overset{AB}{F}^k + D^k/D^3}{R} + \frac{\overset{BA}{F}^k - D^k/D^3}{R} = 0,$$

where $D^k = \overset{A}{\xi}^k - \overset{B}{\xi}^k$.

Newton's law of attraction directly follows.

(15) V. A. FOCK: loc. cit., p. 385.

5. – Second approximation.

We have

$$\mathfrak{g}_3^{00} = \mathfrak{h}_3^{00} = 4 \sum \left(\frac{m}{R} + \varepsilon \dot{m} \right) = 4 \sum \frac{m}{R},$$

$$\mathfrak{g}_4^{0k} = \mathfrak{h}_4^{0k} = 4 \sum \left[\frac{mv^k}{R} + \varepsilon (m\dot{v}^k) \right] = 4 \sum \frac{mv^k}{R},$$

because of $\dot{m} = 0$ and $\sum ma^k = 0$. (This latter relation follows from the just found value of a^k). Application of (3) gives $\sum (\dot{m}/R) = 0$ whence it follows that \dot{m} is a constant, which can be introduced into m . We thus may take, without loss of generality, $\dot{m} = 0$, whence $\mathfrak{g}_3^{00} = 0$ and $\mathfrak{g}_4^{0k} = 0$.

The next step necessitates the computation of $\Theta_{\mathfrak{5}}^{\mu\nu}$. We get

$$\left\{ \begin{array}{l} \Theta_{\mathfrak{5}}^{00} = 0, \\ \Theta_{\mathfrak{5}}^{0k} = \mathfrak{g}_{\mathfrak{2}}^{00,n} (\mathfrak{g}_{\mathfrak{3}}^{0k,n} - \mathfrak{g}_{\mathfrak{3}}^{0n,k}) - \frac{3}{4} \mathfrak{g}_{\mathfrak{2}}^{00,0} \mathfrak{g}_{\mathfrak{2}}^{00,k}, \\ \Theta_{\mathfrak{5}}^{kl} = 0. \end{array} \right.$$

Thus

$$\mathfrak{g}_{\mathfrak{5}}^{kl} = \mathfrak{h}_{\mathfrak{5}}^{kl} = 4\varepsilon \sum m(v^k a^l + v^l a^k).$$

This is a function of time only: $\mathfrak{g}_{\mathfrak{5}}^{kl,t} = 0$. It does not follow, however, that (4) holds identically, because in the course of the first approximation we have made

$$\mathfrak{g}_3^{0k,0} = 4 \sum m \left[v^k \left(\frac{i}{R} \right) + \frac{\dot{v}^k}{R} \right] \rightarrow 4 \sum m \left[v^k \left(\frac{i}{R} \right) + \frac{a^k}{R} \right].$$

It therefore remains, for the second approximation $4 \sum (ma^k/R) = 0$, whence $a_1^k = 0$.

6. – Third approximation.

We have

$$\mathfrak{h}_4^{00} = 4 \sum \left(\frac{m}{R} + \frac{m}{2} \ddot{R} \right),$$

$$\mathfrak{h}_5^{0k} = 4 \sum \left[\frac{mv^k}{R} + \frac{m}{2} (\ddot{v}^k R) \right],$$

and by a method similar to Fock's (15) we get

$$\frac{\tilde{g}^{00}}{4} = 7 \sum \frac{m^2}{R^2} + 14 \sum'_{A,B} \overset{A}{m} \overset{B}{m} S^{ll},$$

$$\frac{\tilde{g}^{0k}}{5} = \frac{1}{2} \sum m^2 \left[15 \frac{v^k}{R^2} - v^l \frac{\partial}{\partial \xi^l} \frac{\partial}{\partial \xi^k} (\log R) \right] + 16 \sum'_{A,B} \overset{A}{m} \overset{B}{m} \left(\frac{3}{4} \overset{A}{v}{}^l S^{lk} + \overset{B}{v}{}^k S^{ll} - \overset{B}{v}{}^l S^{lk} \right).$$

Application of (3) gives

$$\sum \frac{\dot{m}}{R} + \sum'_{A,B} \overset{A}{m} \overset{B}{m} \left(\frac{7}{2} \dot{S}^{ll} + 3 \overset{A}{v}{}^l S^{lk}{}_{,k} + 4 \overset{B}{v}{}^k S^{ll}{}_{,k} - 4 \overset{B}{v}{}^l S^{lk}{}_{,k} \right) = 0.$$

We tentatively try a solution such as

$$\overset{A}{\dot{m}}_2 = \overset{A}{m} \sum'_{A,B} \overset{B}{m} \overset{AB}{K}$$

and equate to zero the coefficient of $\overset{A}{m} \overset{B}{m}$. This gives

$$\frac{\overset{AB}{K}}{\overset{A}{R}} + \frac{(4\overset{A}{v}{}^k - 3\overset{B}{v}{}^k)(D^k/D^3)}{\overset{A}{R}} + \frac{\overset{BA}{K}}{\overset{B}{R}} - \frac{(4\overset{B}{v}{}^k - 3\overset{A}{v}{}^k)(D^k/D^3)}{\overset{B}{R}} = 0.$$

This result, together with Newton's law of attraction which has just been derived, leads to

$$\overset{A}{\dot{m}}_2 = \overset{A}{m} \left(\frac{1}{2} \overset{A}{v}{}^k \overset{A}{v}{}^k + 3 \sum'_{B} \frac{\overset{B}{m}}{D} \right)$$

This is the second-order correction to the « effective source » $M = M_0(dt/d\sigma)g$. The correction to the rest mass M_0 is then found to be (*)

$$\overset{A}{\dot{m}}_0 = \overset{A}{m}_0 \sum'_{B} \frac{\overset{B}{m}_0}{D}.$$

(*) At this approximation $g = 1 + \frac{1}{2} g^{00} = 1 + 2 \sum (m/R)$. As this is computed at the point $\overset{A}{R} = 0$, we have to omit the infinity self-energy of the particle A. This can be done in a consistent way by the use of « good » δ functions (16).

(16) L. INFELD and J. PLEBAŃSKI: *Bull. Acad. Pol. Se., Cl. III.* **4**, 689 (1956); **5**, 51 (1957).

The rest mass is thus increased by the presence of other bodies, a result that has already been found, on quite other grounds, by EINSTEIN (17).

The next step is to compute $\overset{6}{\Theta}^{\mu\nu}$ in order to get $\overset{6}{g}^{kl}$. We find that $\overset{6}{\Theta}^{0k}=0$, but $\overset{6}{\Theta}^{00}$ and $\overset{6}{\Theta}^{kl}$ are extremely cumbersome expressions. We shall therefore postpone the evaluation of $\overset{2}{g}^k$ to the following paper, in which we develop a more powerful method, enabling us to obtain the laws of motion with a much smaller amount of computational labour. This will be, however, at the expense of mathematical elegance and logical simplicity.

7. – Fourth approximation.

As $\overset{5}{\Theta}^{00} = \overset{6}{\Theta}^{0k} = 0$, we should have

$$\overset{5}{g}^{00} = \overset{6}{h}^{00} = 4 \sum \left[\frac{m}{R} + \varepsilon \overset{2}{m} + \frac{\varepsilon}{3!} m(R^2) \right],$$

$$\overset{5}{g}^{0k} = \overset{6}{h}^{0k} = 4 \sum \left[\frac{mv^k}{R} + \varepsilon(\overset{2}{m}v^k) + \frac{\varepsilon}{3!} m(v^k R^2) + \varepsilon m \overset{2}{a}^k \right].$$

However, application of (3) leads to

$$\sum \frac{\overset{3}{m}}{R} + \varepsilon \sum \overset{2}{m} = 0,$$

which implies both $\overset{3}{m} = \text{const} = 0$, and $\sum \overset{2}{m} = 0$. This last relation is not, in general, satisfied. We thus have to add to $\overset{6}{h}^{0k}$ a term such as

$$\overset{6}{S}^{0k} = -\frac{4}{3}\varepsilon \sum (\overset{2}{m} \ddot{R}^k).$$

(Note that $\nabla^2 \overset{6}{S}^{0k} = 0$.) That is, we have to introduce a term which is linear in the sources, although it does not belong to the « linear approximation ». This result may cast some doubt on the validity of the latter.

As most computations on gravitational radiation by a system of bodies are based on this linear approximation (11), it seems to us necessary to pursue

(17) A. EINSTEIN: *The Meaning of Relativity* (Princeton, 1953), p. 102.

the approximation procedure until we find the correction to the acceleration due to radiation reaction. This will be the subject of a later publication.

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I am indebted to Professor NATHAN ROSEN for stimulating discussions.

RIASSUNTO (*)

Si elabora un metodo per risolvere le equazioni di Einstein per approssimazioni successive, l'unicità e la compatibilità essendo assicurate ad ogni passaggio imponendo al campo la condizione di de Donder. Ciò equivale a restringere la libertà di scelta delle sorgenti in modo da determinare le loro leggi di moto direttamente dalle grandezze di campo senza far alcun uso di integrazioni intorno a singolarità. Se il parametro d'approssimazione è il reciproco della velocità della luce, dalla prima approssimazione non banale risulta la costanza della massa a riposo e la legge d'attrazione di Newton. Il passo successivo dimostra che non esistono correzioni del prim'ordine di queste leggi. La terza approssimazione dà poi una piccola correzione della massa a riposo, che si trova aumentata della presenza di altri corpi. C'è anche una correzione del second'ordine dell'accelerazione (qui non calcolata esplicitamente) che provoca la precessione del perielio nel problema dei due corpi. La quarta approssimazione comprende un contributo del campo radiativo. Si trova che per soddisfare la condizione di de Donder dobbiamo introdurre nell'espressione del campo un termine lineare nelle sorgenti per quanto esso non appaia nella cosiddetta « approssimazione lineare ». Quest'ultimo risultato può far sorgere qualche dubbio sulla validità di tale approssimazione. In un ulteriore lavoro risolveremo approssimazioni superiori per mezzo di una tecnica perfezionata e troveremo che la correzione del quint'ordine dell'accelerazione comprende un termine non conservativo. Quest'ultimo risultato può essere ritenuto una prova della realtà della radiazione gravitazionale.

(*) Traduzione a cura della Redazione.

Potential Energy Matrix Elements Between Non-Overlapping Wave Functions (*).

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(ricevuto il 15 Ottobre 1958)

Summary. — Matrix elements of the potential energy of interaction are calculated for the case of two spherically symmetric, localized and non-overlapping wave functions. The result is expressed as a power series, and is applicable to a wide class of central force potential energies, and non-overlapping wave functions. As an illustration, a discussion is given of the diagonal matrix elements of the potential energy corresponding to a Van der Waals law of force.

1. — Introduction.

In many problems one needs to evaluate potential energy matrix elements, $V_{\alpha\beta}(\mathbf{P}, \mathbf{Q})$, between two wave functions, φ_α , φ_β , localized around points \mathbf{P} and \mathbf{Q} , i.e.,

$$(1) \quad V_{\alpha\beta}(\mathbf{P}, \mathbf{Q}) = \iiint \varphi_\alpha^*(\mathbf{x}_1 - \mathbf{P}) \varphi_\beta^*(\mathbf{x}_2 - \mathbf{Q}) v(\mathbf{x}_1, \mathbf{x}_2) \varphi_\alpha(\mathbf{x}_1 - \mathbf{P}) \varphi_\beta(\mathbf{x}_2 - \mathbf{Q}) d^3x_1 d^3x_2 .$$

In most cases of interest, the potential energy function, $v(\mathbf{x}_1, \mathbf{x}_2)$, depends only on the scalar $|\mathbf{x}_1 - \mathbf{x}_2|$, i.e., depends only on the distance between the two

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points, and the φ 's are spherically symmetric functions of their arguments. In this case we may write:

$$(2) \quad V_{\alpha\beta}(\mathbf{P}, \mathbf{Q}) = \iint f_\alpha(r_1) f_\beta(r_2) v(r) d^3r_1 d^3r_2,$$

where

$$r_1 \equiv |\mathbf{x}_1 - \mathbf{P}|, \quad r_2 \equiv |\mathbf{x}_2 - \mathbf{Q}|, \quad r \equiv |\mathbf{x}_1 - \mathbf{x}_2|, \quad f_\alpha(r_1) \equiv \varphi_\alpha^*(r_1) \varphi_\alpha(r_1),$$

and $f_\beta(r_2) \equiv \varphi_\beta^*(r_2) \varphi_\beta(r_2)$.

In the following we discuss a general method, suitable for numerical calculations, to evaluate the six dimensional integrals for $V_{\alpha\beta}(\mathbf{P}, \mathbf{Q})$ for a wide class of potential energies of interaction and angle-independent, localized non-overlapping wave functions, *i.e.*, to evaluate these integrals with: $f_\alpha(r_1) = 0$ when $r_1 > a$, $f_\beta(r_2) = 0$ when $r_2 > b$, and $(a+b) < |\mathbf{P} - \mathbf{Q}| = R$. Our results for $V_{\alpha\beta}(\mathbf{P}, \mathbf{Q})$, Eqs. (6a), (6b) below, will be expressed as a power series in terms of a/R and b/R , and will be found to involve the potential energy function $v(r)$ and its derivatives, and the moments of the probability densities $f_\alpha(r_1)$, $f_\beta(r_2)$. We shall also discuss the rate of convergence of the series, which depends on the functional form of $v(r)$ and on the ratios a/R and b/R .

2. – Integration.

The integral yielding the matrix element under discussion is

$$(3) \quad V_{\alpha\beta}(R, a, b) = \iint_{\Omega_1 \Omega_2} f_\alpha(r_1) f_\beta(r_2) v(r) d^3r_1 d^3r_2,$$

where Ω_1 and Ω_2 are respectively two non-overlapping spheres of radii a and b , with centers at points \mathbf{P} and \mathbf{Q} .

We may write:

$$(4) \quad V_{\alpha\beta}(R, a, b) = \int_{\Omega_2} f_\beta(r_2) V_1(z, a) d^3r_2,$$

where

$$V_1(z, a) = \int_{\Omega_1} f_\alpha(r_1) v(r) d^3r_1,$$

the meaning of the symbols being given in Fig. 1.

In order to calculate $V_1(z, a)$ we first integrate

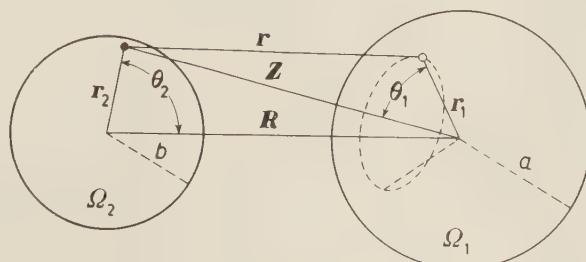


Fig. 1.

over the rim of the cone: $r_1 = \text{const.}$, $\theta_1 = \text{const.}$, and obtain:

$$V_1(z, a) = 2\pi \int_0^a \int_0^\pi f_\alpha(r_1) v(r) r_1^2 \sin \theta_1 d\theta_1 dr_1.$$

Next we change the variable of integration from θ_1 to r by means of the relation:

$$r^2 = z^2 + r_1^2 - 2zr_1 \cos \theta_1,$$

so that

$$V_1(z, a) = \frac{2\pi}{z} \int_0^a r_1 f_\alpha(r_1) dr_1 \int_{z-r_1}^{z+r_1} w(r) dr = \frac{2\pi}{z} \int_0^a [W(z+r_1) - W(z-r_1)] r_1 f_\alpha(r_1) dr_1,$$

where by definition:

$$w(r) \equiv rv(r)$$

and

$$W(x) \equiv \int_t^x w(r) dr = \int_t^x rv(r) dr,$$

t being an arbitrary number ($t \neq x$) whose value does not have any effect on $V_1(z, a)$.

Any $v(r)$ of physical interest is such that $W(x)$ is a function which can be expanded into a uniformly convergent power series and hence can be integrated term by term. These two steps lead to

$$(5) \quad V_1(z, a) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\lambda_\alpha^{(2n)}(a)}{(2n+1)!} w^{(2n)}(z),$$

where

$$w^{(2n)}(z) \equiv \left[\frac{d^{2n}}{dy^{2n}} w(y) \right]_{y=z}; \quad w^{(0)}(z) = w(z),$$

and

$$\lambda_\alpha^{(2n)}(a) \equiv \int_{\Omega_s} r_1^{2n} f_\alpha(r_1) d^3 r_1; \quad \lambda_\alpha^{(0)}(a) = 1,$$

is the $(2n)$ -th moment of the probability density around point P .

Eq. (5) is a series of even powers of a , and represents the potential energy produced at a point z exterior to Ω_1 by a particle whose wave function φ_α is effectively confined to the interior of Ω_1 .

Combining Eqs. (5) and (4) and changing the variable of integration in Eq. (3) from θ_2 to z , yields:

$$V_{\alpha\beta}(R, a, b) = \frac{2\pi}{R} \sum_{n=0}^{\infty} \frac{\lambda_x^{(2n)}(a)}{(2n+1)!} \int_0^b [U(R' + r_2) - U(R - r_2)] r_2 f_\beta(r_2) dr_2 ,$$

where

$$U(x) \equiv \int_i^x w^{(2n)}(z) dz ,$$

so that expanding $U(x)$ in a power series, and integrating term by term, we finally get:

$$(6a) \quad V_{\alpha\beta}(R, a, b) = \frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda_x^{(2n)}(a) \lambda_\beta^{(2m)}(b)}{(2n+1)! (2m+1)!} w^{(2(n+m))}(R) ,$$

where

$$w^{(2(n+m))}(R) \equiv \left[\frac{d^{2(n+m)}}{dy^{2(n+m)}} w(y) \right]_{y=R} ,$$

and

$$\lambda_\beta^{(2m)}(b) \equiv \int_{\Omega_2} r_2^{2m} f_\beta(r_2) d^3 r_2 ,$$

is the $(2m)$ -th moment of the probability density around point Q .

Eq. (6a) represents the desired matrix element. Its meaning becomes more transparent if we separate out the first term in the series, *i.e.*, the term with $m = 0 = n$. We obtain

$$(6b) \quad V_{\alpha\beta}(R, a, b) = v(R) \left\{ 1 + \sum_{n=0}^{\infty} \sum_{\substack{m=0 \\ n+m \neq 0}}^{\infty} \frac{\lambda_x^{(2n)}(a) \lambda_\beta^{(2m)}(b)}{(2n+1)! (2m+1)!} \frac{w^{(2(n+m))}(R)}{w(R)} \right\} .$$

The first term, $v(R) \cdot 1$, in Eq. (6b) represents the classical potential energy of interaction of two point sources located at points P and Q . The remainder term, $v(R) \cdot \sum_n \sum_m \dots$, is either a polynomial or a power series in a/r , b/R , which in either case vanishes in the limit $a \rightarrow 0$, $b \rightarrow 0$. The coefficients in this term can be evaluated in a straightforward way once $v(r)$ and the wave functions $\varphi_\alpha(r_1)$, $\varphi_\beta(r_2)$ are known.

In the particular case of diagonal matrix elements, $\alpha = \beta$, and if $a = b$ we can rearrange the terms in Eq. (6b) in the following way

$$(7) \quad V_{\alpha\alpha}(R, a) = v(R) \left\{ 1 + (a/R)^2 [T_{\alpha}^{(01)} + T_{\alpha}^{(10)}] + (a/R)^4 [T_{\alpha}^{(02)} + T_{\alpha}^{(11)} + T_{\alpha}^{(20)}] + (a/R)^6 [T_{\alpha}^{(03)} + T_{\alpha}^{(12)} + T_{\alpha}^{(21)} + T_{\alpha}^{(30)}] + \dots \right\},$$

where

$$(8) \quad T_{\alpha}^{st} = (R/a)^{2(s+t)} \frac{\lambda_{\alpha}^{2s}(a) \lambda_{\alpha}^{2t}(a)}{(2s+1)! (2t+1)!} \cdot \frac{w^{[2(s+t)]}(R)}{w(R)}.$$

3. – Illustrative examples.

The terms inside the bracket in Eq. (6b) may contain a finite or an infinite number of terms, depending on the functional form of $v(r)$:

- a) For a Coulomb inverse square law of force, $v(r) \sim r^{-1}$, and $w(r) = \text{const.}$; the remainder term then vanishes identically and $V_{\alpha\beta}(R, a, b) = v(R)$ in agreement with the well known result of classical electrostatics.
- b) If $v(r)$ is a polynomial of (p) -th degree in r , $p > 0$, then our series in Eq. (6b) or (7), reduces to a polynomial of the same degree. In this case the evaluation of the right hand side of Eq. (6b) or (8) is in general less tedious than a direct evaluation of the integral in Eq. (2).
- c) Finally, for all other $v(r)$, the series in Eq. (6b) does not terminate but none-the less converges for $(a+b) < R$. The rate of convergence depends on the values of the space derivatives of the potential $v(r)$, evaluated at $r = R$.

In order to see the usefulness of our expansion for a $v(r)$ of type c), we, in a particular case, compare approximate values of $V_{\alpha\alpha}(R, a)$ with the corresponding exact values. The integral in Eq. (2) for $V_{\alpha\alpha}(R, a)$ has been evaluated in closed form by HADAMAKER⁽¹⁾ for a Van der Waals potential energy $v(r) = Kr^{-6}$, ($K = \text{const.}$) and with probability densities

$$f_{\alpha}(r) = f_{\beta}(r) = (4\pi a^3/3)^{-1}, \quad \text{for } r \leq a,$$

and

$$f_{\alpha}(r) = f_{\beta}(r) = 0 \quad \text{for } r > a.$$

⁽¹⁾ H. C. HADAMAKER: *Physica*, **4**, 1058 (1937).

With these probability densities the moments $\gamma_{\alpha}^{(2n)}(a) = 3a^{2n}/(2n+3)$, so that Eqs. (7) and (8) yield:

$$(9) \quad V_{\alpha\alpha}(R, a) = KR^{-6} \left\{ 1 + (6/4)x^2 + (9/5)x^4 + \dots + 3(n+1)/(n+3) \cdot x^{2n} + \dots \right\},$$

where $x = 2a/R$. On the other hand, the exact result found by HADAMAKER is

$$(10) \quad V_{\alpha\alpha}(R, a) = \frac{3K}{32a^6} \left\{ \frac{2a^2}{R^2 - 4a^2} + \frac{2a^2}{R^2} + \ln \left(1 - \frac{4a^2}{R^2} \right) \right\}.$$

In Fig. 2 we have plotted, for various values of $2a/R$, the exact value of $V_{\alpha\alpha}(R, a)$ as given by Eq. (10), (curve *H*), and our result for $V_{\alpha\alpha}(R, a)$ in Eq. (9) including terms up to x^6 (curve *A*), and x^8 (curve *B*). Table I illustrates the accuracy of our approximation.

From Fig. 2 and Table I we can conclude that our series for $V_{\alpha\alpha}(R, a)$ converges rapidly for $2a/R < 0.5$ in the case of Van der Waals forces. For potential energies of the form $v(r) \sim r^{-p}$ ($p > 1$), this series converges more rapidly for $p < 6$, than for $p > 6$. In a paper to appear elsewhere (2), the author has applied the present method, using Lennard-Jones « 12-6 » potential energy functions and various probability density functions, to study the binding energy of crystals at low temperatures.

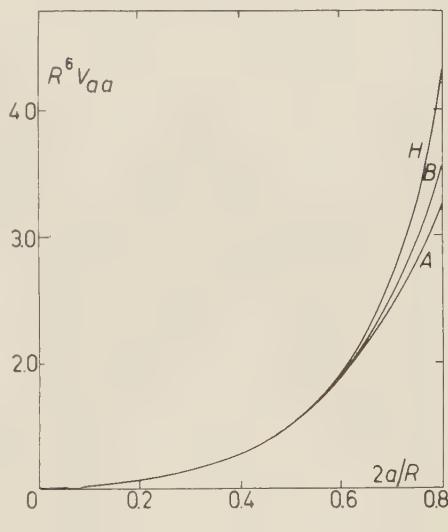


Fig. 2.

TABLE I.

$2a/R$	ΔA (%)	ΔB (%)
0.2	0.01	< 0.01
0.4	0.25	0.20
0.6	2	1
0.8	25	16

(2) N. BERNARDES: to appear in *Phys. Rev.* (1958).

* * *

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R I A S S U N T O (*)

Si calcolano gli elementi di matrice dell'energia potenziale di interazione per il caso di due funzioni d'onda a simmetria sferica, localizzate e non sovrapposte. Il risultato si esprime in serie di potenze ed è applicabile ad una vasta classe di energie potenziali di forze centrali e di funzioni d'onda non sovrapposte. A titolo illustrativo, si dà una discussione degli elementi di matrici diagonali dell'energia potenziale corrispondenti a una forza di Van der Waals.

(*) Traduzione a cura della Redazione.

**The Partition Function
and the Subsidiary Conditions in the Bohm-Pines
and Migdal-Galitsky's Supplementary Variables Method.**

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(ricevuto il 31 Ottobre 1958)

Summary. — The supplementary variables method is applied to the approximate calculation of the partition function of N interacting fermions. It is shown that the supplementary variables method in its original formulation due to Bohm-Pines and Migdal-Galitsky leads to wrong results. One obtains correct results if one assumes a modification, due to Kanazawa, of the subsidiary conditions in this method.

1. — Introduction.

In this work (*) we calculate approximately the partition function of a N -body system using the supplementary (collective) variables method of Bohm-Pines (¹) and Migdal-Galitsky (²) (hereafter quoted as BPMG). In this method the subsidiary conditions which are imposed on the state vectors of the system play an important role. Recently KANAZAWA (³) has pointed out that the subsidiary conditions of BPMG must be modified analogously to the modification of the Lorentz condition in quantum electrodynamics due to GUPTA and BLEULER (⁴). The necessity of this modification results also from our work:

(*) Some of results obtained here were published as a short note in *Dokl. Akad. Nauk SSSR*, **118**, 61 (1958).

(¹) D. PINES: *Solid state physics*, **1**, 368 (1955).

(²) D. N. ZUBAREV: *Candidate's degree dissertation* (Moscow, 1953).

(³) H. KANAZAWA: *Progr. Theor. Phys.*, **18**, 287 (1957).

(⁴) S. N. GUPTA: *Proc. Phys. Soc.*, A **63**, 681 (1950); K. BLEULER: *Helv. Phys. Acta*, **23**, 567 (1950).

only the partition function calculated with the use of the modified subsidiary conditions fulfils the correspondence principle for $\hbar \rightarrow 0$.

In the calculations we use a discrete parameter α which takes two values: 0 and 1. It enables us to consider simultaneously both formulations of the subsidiary conditions: the original and the modified one.

2. – The supplementary variables method.

In this Section we follow the papers (1-3).

Let us consider a system of N fermions (the case of bosons can be treated quite analogously) contained in a volume v interacting through a two-body central potential $W_{ij} = W(|\mathbf{x}_i - \mathbf{x}_j|)$. The Hamiltonian of the system is

$$(1) \quad \hat{\mathcal{H}} = \hat{T} + \hat{\mathcal{O}} \quad (*).$$

Here

$$(2) \quad \hat{T} = \sum_{i=1}^N \frac{\hat{p}_i^2}{2m},$$

$$(3) \quad \hat{\mathcal{O}} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \hat{W}_{ij} = \frac{N}{2v} \sum'_{\mathbf{k}} W_{\mathbf{k}} \hat{Q}_{\mathbf{k}} \hat{Q}_{-\mathbf{k}} + \frac{N^2}{2v} W_0 - \frac{N}{2v} \sum_{\mathbf{k}} W_{\mathbf{k}} \quad (**),$$

where

$$W_{\mathbf{k}} = \int W(|\mathbf{x}|) \exp[-i\mathbf{k}\mathbf{x}] d^3x,$$

$$\varrho_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{j=1}^N \exp[-i\mathbf{k}\mathbf{x}_j].$$

By $\hat{Q}_{\mathbf{k}}^R$ and $\hat{Q}_{\mathbf{k}}^I$ we shall denote the hermitian and the antihermitian parts of the operator $\sqrt{2} \hat{Q}_{\mathbf{k}}$, respectively:

$$(4) \quad \hat{Q}_{\mathbf{k}} = \frac{1}{\sqrt{2}} (\hat{Q}_{\mathbf{k}}^R + i\hat{Q}_{\mathbf{k}}^I).$$

We now extend the above system by adding some new degrees of freedom described by variables $\hat{Q}_{\mathbf{k}}$ (the supplementary variables), ($\mathbf{k} \neq 0$ and $|\mathbf{k}| < k_0$,

(*) \hat{B} denotes an operator and B his eigenvalue.

(**) \sum' denotes sum over \mathbf{k} with omitting the term $\mathbf{k} = 0$.

k_0 is some constant). For these variables we have, similarly as for $\hat{\varrho}_k$,

$$(5) \quad \hat{Q}_k^+ = \hat{Q}_{-k}$$

and

$$(6) \quad \hat{\varrho}_k = \frac{1}{\sqrt{2}} (\hat{Q}_k^R + i\hat{Q}_k^I).$$

We define the extended Hamiltonian as follows

$$(7) \quad \hat{H}_\alpha = \hat{T} + \hat{V}_\alpha, \quad (\alpha = 1, 0),$$

where

$$(8) \quad \hat{V}_\alpha = \frac{N}{2v} \sum_{|k|>k_0} W_k \hat{\varrho}_k \hat{\varrho}_{-k} + \frac{N}{2v} \sum'_{|k|>k_0} W_k (\hat{\varrho}_k + \hat{Q}_k) (\hat{\varrho}_{-k} + \hat{Q}_{-k}) + A_\alpha,$$

and

$$(9) \quad A_\alpha = \frac{N^2}{2v} W_0 - \frac{N}{2v} \sum_k W_k - \frac{\alpha}{4} \sum'_{|k|<k_0} \hbar \omega_k,$$

$$(10) \quad \omega_k = \sqrt{\frac{1}{m} \frac{N}{v} k^2 W_k}.$$

We perform now an unitary transformation making use of the operator

$$(11) \quad \hat{U} = \exp \left\{ -\frac{i}{\hbar} \sum'_{|k|<k_0} P_k \hat{\varrho}_k \right\},$$

where $\hat{P}_k = (1/\sqrt{2})(\hat{P}_k^R + i\hat{P}_k^I)$ is the momentum canonically conjugate to $\hat{\varrho}_k$:

$$(12) \quad [\hat{\varrho}_k, \hat{P}_k] = i\hbar.$$

After performing the transformation (11) we have

$$(13) \quad \hat{H}'_\alpha = \hat{U} \hat{H}_\alpha \hat{U}^{-1} = \hat{T} + \hat{H}_{\text{osc}} + A_\alpha + \hat{H}_I,$$

where

$$(14) \quad \hat{H}_{\text{osc}} = \sum'_{|k|<k_0} \hat{H}_{\text{osc} k} = \sum'_{|k|<k_0} \left(\frac{1}{2m} k^2 \hat{P}_k \hat{P}_{-k} + \frac{N}{2v} W_k \hat{\varrho}_k \hat{\varrho}_{-k} \right) = \frac{1}{2} \sum'_{|k|<k_0} (\hat{H}_{\text{osc} k}^R + \hat{H}_{\text{osc} k}^I),$$

is the Hamiltonian of many linear oscillators vibrating with the frequencies ω_k , and \hat{H}_I contains terms describing the interaction between the fermions, between the oscillators, and between the fermions and the oscillators (1,2).

The extended system with the Hamiltonian (7) must be equivalent to the original system. Therefore, it is necessary to impose some subsidiary conditions on the state vectors of the extended system:

$$(15) \quad \hat{L}_{k\alpha} |\rangle_{\alpha} = 0 \quad |k| < k_0,$$

where

$$(16) \quad \hat{L}_{k\alpha} = \hat{Q}_k + \frac{i\alpha}{\gamma_k^2} \hat{P}_k, \quad \gamma_k = \sqrt[4]{m \frac{N}{v} \frac{W_k}{k^2}}.$$

After the transformation (11) the subsidiary conditions become

$$(17) \quad \hat{L}'_{k\alpha} |\rangle'_{\alpha} = 0 \quad |k| < k_0,$$

where

$$(18) \quad \hat{L}'_{k\alpha} = \hat{U} \hat{L}_{k\alpha} \hat{U}^{-1} = \hat{L}_{k\alpha} - \hat{Q}_k$$

and

$$(19) \quad |\rangle'_{\alpha} = \hat{U} |\rangle_{\alpha}.$$

It is easy to verify that the substitution $\alpha=0$ in the above expressions yields the original formulation of the supplementary variables theory due to BPMG, and the substitution $\alpha=1$ yields the modified formulation due to KANAZAWA (the method b) of (3)).

3. – The calculation of the partition function.

The partition function Z_N of the system of N fermions is

$$(20) \quad Z_N = \mathcal{C}\text{Tr}(\exp[-\beta \hat{\mathcal{H}}]) = \text{Tr}(\exp[-\beta H_N]),$$

where $\beta = 1/kT$ and $\mathcal{C}\text{Tr}$ denotes the trace over a complete set of states of the original system, and Tr that over a complete set of states of the extended system satisfying the subsidiary conditions (15). The trace is an invariant of the unitary transformations, and, therefore, we have also

$$(21) \quad Z_N = \text{Tr}'(\exp[-\beta \hat{H}'_{\alpha}]),$$

where Tr' denotes the trace over a complete set of states of the extended system satisfying the transformed subsidiary conditions (17).

We shall calculate Z_N in an approximation in which we neglect the term \hat{H}'_{α} in the exponent.

We now build up the complete set of the states over which we sum in (20). Let us note that \hat{L}_{k_α} depends only on the supplementary variables. Therefore, the states of the system satisfying the conditions (15) may be assumed as follows

$$(22) \quad |xb\rangle_\alpha = |x\rangle |b\rangle_\alpha,$$

where $|x\rangle$ is the common eigenvector of the position operators of N particles $\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_N$:

$$(23) \quad |x\rangle = \prod_{j=1}^N |\mathbf{x}_j\rangle,$$

and

$$(24) \quad |b\rangle_\alpha = \prod'_{|\mathbf{k}| < k_0} |b_{\mathbf{k}}\rangle_\alpha = \prod'_{|\mathbf{k}| < k_0} |b_{\mathbf{k}}^R\rangle_\alpha |b_{\mathbf{k}}^I\rangle_\alpha \quad (*)$$

is a vector describing the oscillators. The vectors $|b_{\mathbf{k}}\rangle$ satisfy the conditions

$$(25) \quad \hat{L}_{\mathbf{k}}, b_{\mathbf{k}} = 0, \quad |\mathbf{k}| < k_0.$$

We take such vectors $|x\rangle$ in (22) because they are convenient with respect to the form of the operator \hat{U} .

In further calculations we shall use the common eigenvectors of the operators $\hat{Q}_{\mathbf{k}}$ and we assume for them the notation

$$(26) \quad |Q\rangle = \prod'_{|\mathbf{k}| < k_0} |Q_{\mathbf{k}}\rangle = \prod'_{|\mathbf{k}| < k_0} |Q_{\mathbf{k}}^R\rangle |Q_{\mathbf{k}}^I\rangle.$$

We perform now the transformation (11) and after simple calculations we obtain

$$(27) \quad |xb'\rangle_\alpha = \hat{U} |xb\rangle_\alpha = \int |x\rangle |Q\rangle dQ \langle Q - \varrho(x) |b\rangle_\alpha,$$

where

$$dQ = \prod'_{|\mathbf{k}| < k_0} dQ_{\mathbf{k}}^R dQ_{\mathbf{k}}^I.$$

We see then that

$$(28) \quad \text{Tr}'(\dots) = \int dx \langle xb | \dots | xb' \rangle_\alpha,$$

(*) \prod'_k denotes the product over \mathbf{k} ($\mathbf{k} \neq 0$) lying in a hemisphere (because $Q_{-\mathbf{k}}^R = Q_{\mathbf{k}}^R$, $Q_{-\mathbf{k}}^I = -Q_{\mathbf{k}}^I$).

where

$$dx = \prod_{i=1}^N d^3x_i.$$

Therefore

$$(29) \quad Z_{N\alpha}^0 = \text{Tr}'(\exp[-\beta(\hat{T} + \hat{H}_{\text{osc}} + A_\alpha)]) = \exp[-\beta A_\alpha] \int dx \langle x | \exp[-\beta \hat{T}] | x \rangle \cdot \\ \cdot \int dQ dQ' G_\alpha^*(Q' - \varrho(x)) G_\alpha(Q - \varrho(x)) \langle Q' | \exp[-\beta H_{\text{osc}}] | Q \rangle,$$

where

$$(30) \quad G_\alpha(Q) = \prod_{|k| < k_0}'' g_{k\alpha}(Q_k^R) g_{k\alpha}(Q_k^I),$$

$$(31) \quad g_{k\alpha}(Q_k^R) = \langle Q_k^R | b_k^R \rangle_\alpha, \quad g_{k\alpha}(Q_k^I) = \langle Q_k^I | b_k^I \rangle_\alpha.$$

From (25) we get

$$(32) \quad g_{k\alpha}(q) = S_{k\alpha} \exp \left[-\frac{1}{2\alpha} \left(\frac{q}{C_k} \right)^2 \right],$$

where

$$(33) \quad S_{k\alpha} = \frac{1}{(\sqrt{2}\sqrt{\pi}C_k)^{1-\alpha}} \frac{1}{\sqrt{\alpha}C_k\sqrt{\pi}},$$

$$(34) \quad C_k = \frac{\sqrt{h}}{\gamma_k}.$$

Thus, we have

$$(35) \quad g_{k0}(q) = \delta(q), \quad (*)$$

and

$$(36) \quad g_{k1}(q) = \frac{1}{\sqrt{\sqrt{\pi}C_k}} \exp \left[-\frac{1}{2} \left(\frac{q}{C_k} \right)^2 \right].$$

For the mean values of the operator $\exp[-\beta \hat{T}]$ we have

$$(37) \quad \langle x | \exp[-\beta \hat{T}] | x \rangle = \left(\frac{1}{N!} \right)^2 v^{-N} \sum_{f_1} \dots \sum_{f_N} \exp \left[-\beta \sum_{j=1}^N \varepsilon_{f_j} \right] |\det(\exp[-if_a x_b])|^2,$$

(*) We treat g_{k0} as $\lim_{\alpha \rightarrow 0} g_{k\alpha}$ assuming here that α is a continuous parameter.

where

$$\varepsilon_i = \frac{\hbar^2 f_i^2}{2m}.$$

The matrix elements $\langle Q' | \exp[-\beta \hat{H}_{osc}] | Q \rangle$ have the following structure

$$(38) \quad \begin{aligned} \langle Q' | \exp[-\beta \hat{H}_{osc}] | Q \rangle &= \\ &= \prod_k'' \langle Q_k^{R'} | \exp[-\beta \hat{H}_{osc k}^R] | Q_k^R \rangle \langle Q_k^T | \exp[-\beta \hat{H}_{osc k}^T] | Q_k^T \rangle . \end{aligned}$$

After short calculations in which we have made use of the summation formula for the Hermite polynomials (5) we obtain

$$(39) \quad \begin{cases} \langle Q_k^{R'} | \exp[-\beta \hat{H}_{osc k}^R] | Q_k^R \rangle = f_k(Q_k^{R'}, Q_k^R) \\ \langle Q_k^T | \exp[-\beta \hat{H}_{osc k}^T] | Q_k^T \rangle = f_k(Q_k^T, Q_k^T), \end{cases}$$

where

$$(40) \quad f_k(q', q) = \frac{1}{C_k} \sqrt{\frac{t_k}{\pi(1-t_k^2)}} \exp \left[\frac{-1}{2C_k^2(1-t_k^2)} [(1+t_k^2)(q'^2 + q^2) - 4t_k q' q] \right],$$

$$t_k = \exp[-\beta \hbar \omega_k].$$

Substituting (32), (37), (39) into the integrand in (29) and performing the integration over $dQ dQ'$, we obtain

$$(41) \quad Z_{N\alpha}^0 = \exp[-\beta A_\alpha] \left(\frac{1}{N!} \right)^2 v^{-N} \sum_{f_1} \dots \sum_{f_N} \exp \left[-\beta \sum_{j=1}^N \varepsilon_{f_j} \right] \cdot$$

$$\cdot \int dx |\det(\exp[-i\mathbf{f}_a \mathbf{x}_b])|^2 \prod_{|\mathbf{k}|<} Y_{kx},$$

where

$$(42) \quad Y_{kx} = 2^{2\alpha} \pi^{\alpha-1} C_k^{2(\alpha-1)} \frac{t_k}{1-t_k^2} \frac{1}{(1+\alpha(1-t_k)/(1+t_k))(1+\alpha(1+t_k)/(1-t_k))} \cdot$$

$$\cdot \exp \left[-\frac{2}{C_k^2} \frac{(1-t_k)/(1+t_k)}{1+\alpha(1-t_k)/(1+t_k)} Q_k Q_{-k} \right].$$

The final expressions for the partition function for both values of α are as

(5) E. C. TITCHMARSH: *Introduction to the Theory of Fourier Integrals* (Oxford, 1948), p. 77, theorem 53.

follows

$$(43) \quad Z_{N\alpha=0}^0 = \exp[-\beta A_0] \left(\frac{1}{N!} \right)^2 v^{-N} \sum_{f_1} \dots \sum_{f_N} \exp \left[-\beta \sum_{j=1}^N \varepsilon_{f_j} \right] \cdot \\ \cdot \int dx |\det(\exp[-i\mathbf{f}_a \mathbf{x}_b])|^2 \prod'_{|\mathbf{k}| < k_0} \frac{1}{\pi} \frac{1}{C_k^2} \frac{t_k}{1-t_k^2} \exp \left[-\frac{2}{C_k^2} \frac{1-t_k}{1+t_k} \varrho_{\mathbf{k}} \varrho_{-\mathbf{k}} \right],$$

$$(44) \quad Z_{N\alpha=1}^0 = \exp[-\beta A_1] \left(\frac{1}{N!} \right)^2 v^{-N} \sum_{f_1} \dots \sum_{f_N} \exp \left[-\beta \sum_{j=1}^N \varepsilon_{f_j} \right] \cdot \\ \cdot \int dx |\det(\exp[-i\mathbf{f}_a \mathbf{x}_b])|^2 \prod'_{|\mathbf{k}| < k_0} t_k \exp \left[-\frac{1}{C_k^2} (1-t_k) \varrho_{\mathbf{k}} \varrho_{-\mathbf{k}} \right].$$

In the next Section we shall show that the correct partition function is given by (44).

4. – The classical limit $\hbar \rightarrow 0$.

We shall now consider the transition to the classical statistics, and, therefore, we shall neglect the fact that the quantum particles are indistinguishable assuming

$$(45) \quad \left(\frac{1}{N!} \right)^2 |\det(\exp[-i\mathbf{f}_a \mathbf{x}_b])|^2 \sim 1.$$

Thus we have

$$(46) \quad Z_{N\alpha}^0 = \sum_{f_1} \dots \sum_{f_N} \exp \left[-\beta \sum_{j=1}^N \varepsilon_{f_j} \right] R_{N\alpha},$$

where

$$(47) \quad R_{N0} = \exp[-\beta A_0] v^{-N} \prod'_{|\mathbf{k}| < k_0} \frac{1}{\pi} \frac{1}{C_k^2} \frac{t_k}{1-t_k^2} \int dx \exp \left[-\frac{2}{C_k^2} \frac{1-t_k}{1+t_k} \varrho_{\mathbf{k}}(x) \varrho_{-\mathbf{k}}(x) \right],$$

and

$$(48) \quad R_{N1} = \exp[-\beta A_1] v^{-N} \prod'_{|\mathbf{k}| < k_0} t_k \int dx \exp \left[\frac{1}{C_k^2} (1-t_k) \varrho_{\mathbf{k}}(x) \varrho_{-\mathbf{k}}(x) \right].$$

It is easy to verify that the classical limit $\hbar \rightarrow 0$ with the assumption $k_0 = \infty$ gives for R_{N1} the expression

$$(49) \quad \lim_{\hbar \rightarrow 0} R_{N1} = \exp \left[-\frac{\beta}{2} \left(\frac{N^2}{v} W_0 - \frac{N}{2v} \sum_k W_k \right) \right] \int dx \exp \left[-\frac{\beta}{2} \frac{N}{v} \sum_k W_k \varrho_{\mathbf{k}} \varrho_{-\mathbf{k}} \right],$$

which is exactly the classical configurational integral « das Konfigurationsintegral »:

$$(50) \quad \int \exp \left[-\frac{\beta}{2} \sum_{\substack{i,j \\ i \neq j}} W_{ij} \right] dx .$$

The classical limit $\hbar \rightarrow 0$ gives for R_{N_0} a divergent expression.

We see, therefore, that the correspondence principle requires the modification of the subsidiary conditions of the method of BPMG in the form proposed by KANAZAWA because the subsidiary conditions in the original form lead to meaningless results.

5. – Further approximation.

The evaluation of the integrals (49) is very complicated and, therefore, we shall perform them approximately. ZUBAREV ⁽⁶⁾ has given a method for the approximate calculation of the classical configurational integral (50) based on the expansion of the integrand in powers of $1/\sqrt{N}$. This method can be applied to the evaluation of R_N (we write R_N for R_{N_1}). We have calculated R_N by this method up to terms $\sim 1/N$ with the following result:

$$(51) \quad R_N = \exp \left[-\beta \left(\frac{N^2}{2v} W_0 - \frac{N}{2v} \sum_{\mathbf{k}} W_k \right) \right] \prod'_{|\mathbf{k}| < k_0} \exp \left[-\frac{\beta \hbar \omega_k}{2} \right] \cdot \left(\frac{1}{C_k^2} (1 - \exp [-\beta \hbar \omega_k]) + 1 \right)^{-1} .$$

⁽⁶⁾ D. N. ZUBAREV: *Dokl. Akad. Nauk SSSR*, **95**, 757 (1954).

RIASSUNTO (*)

Il metodo delle variabili supplementari si applica al calcolo approssimato della funzione di ripartizione di N fermioni interagenti. Si dimostra che il metodo delle variabili supplementari nella sua formulazione originale dovuta a Bohm-Pines e a Migdal-Galitsky conduce a risultati errati. Si ottengono risultati corretti se si introduce una modificazione delle condizioni sussidiarie del metodo dovuta a Kanazawa.

(*) Traduzione a cura della Redazione.

Gravitational Motion and Radiation - II.

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Summary. — General relativistic laws of motion for slow particles are deduced from successive approximate solutions of Einstein's equations, subject to de Donder's conditions. The first three approximations give results identical with those of the previous paper ⁽¹⁾ (Newtonian and post-Newtonian equations of motion). Higher approximations involve contributions from the radiation field. It is found that they can be treated independently of other relativistic effects, and cause fifth-order corrections to the rest-masses and accelerations of the particles. In particular, it is shown that as a consequence of gravitational radiation, a system of particles *gains* energy, thus implying that gravitational waves carry away a *negative* amount of energy. This is made plausible by showing that the energy density of the gravitational field is also a negative quantity. An electromagnetic analogy of these results is proposed.

1. — Equations of motion.

In a previous paper ⁽¹⁾ we chose de Donder's relations as uniqueness and compatibility conditions, in order to solve successive approximations of the $g^{\mu\nu}$ field. It was shown that the laws of motion could then be deduced directly from the field quantities without any use of integration around singularities.

While the principle of this method is extremely simple, its application to higher approximations necessitates an enormous amount of computational labour. If we are interested only in the laws of motion (and not in the $g^{\mu\nu}$ field *per se*), an equivalent ⁽²⁾ and much more speedy method is based on the conservation

⁽¹⁾ A. PERES: *Nuovo Cimento*, **11**, 617 (1959). Hereafter referred to as I. All notations throughout the present paper are those of I, unless otherwise stated.

⁽²⁾ PHAM TAN HOANG: *Compt. Rend.*, **246**, 1497 (1958); *Nuovo Cimento*, **9**, 647 (1958).

laws $\mathfrak{T}^{\mu\nu}_{;\nu} = 0$. As we are dealing with point particles, it is convenient to introduce Infeld's «good» $\hat{\delta}$ function ^(3,4), having the properties:

$$\int f(\mathbf{x}) \hat{\delta}(\mathbf{x} - \xi) dV = f(\xi); \quad \int \frac{f(\mathbf{x}) \hat{\delta}(\mathbf{x} - \xi)}{|\mathbf{x} - \xi|^n} dV = 0,$$

where $f(\mathbf{x})$ is regular at $\mathbf{x} = \xi$.

We have shown, in I, that the «effective source» of the field is the second order tensor density

$$g\mathfrak{T}^{\mu\nu} = \sum M v^\mu v^\nu \hat{\delta}(\mathbf{x} - \xi)$$

rather than the simple tensor density $\mathfrak{T}^{\mu\nu}$. Indeed, we get simpler formulae by writing

$$(1) \quad (g\mathfrak{T}^{\mu\nu})_{;\nu} = (g\mathfrak{T}^{\mu\nu})_{,\nu} + (g\mathfrak{T}^{\alpha\beta}) F_{\alpha\beta}^\mu = 0,$$

where

$$\begin{aligned} F_{\alpha\beta}^\mu &= \Gamma_{\alpha\beta}^\mu - \frac{1}{2} (\delta_{\alpha}^{\mu} \Gamma_{\beta\gamma}^\gamma - \delta_{\beta}^{\mu} \Gamma_{\alpha\gamma}^\gamma) \\ &= \frac{1}{2} [g^{\mu\gamma} (g_{\alpha\pi} g_{\beta\varrho} - \frac{1}{2} g_{\alpha\beta} g_{\pi\varrho}) - \delta_{\alpha}^{\mu} \delta_{\beta}^{\gamma} g_{\gamma\varrho} - \delta_{\pi}^{\mu} \delta_{\alpha}^{\gamma} g_{\beta\varrho}] g^{\pi\varrho}. \end{aligned}$$

Now

$$(g\mathfrak{T}^{\mu\lambda})_\nu = \sum \left[\frac{d}{dt} (M v^\mu \hat{\delta}) - M v^\mu v^k \frac{\partial}{\partial \xi^k} \hat{\delta} \right] = \sum (M \dot{v}^\mu) \hat{\delta}.$$

Thus (1) reads

$$(\dot{M} v^\mu) + M v^\alpha v^\beta \bar{F}_{\alpha\beta}^\mu = 0,$$

where the bar over $\bar{F}_{\alpha\beta}^\mu$ means that we have to compute $F_{\alpha\beta}^\mu$ at the position of the particle which we are dealing with, and to cut-off all infinite self terms ⁽⁴⁾.

Equation (2) gives laws of motion that are perfectly equivalent to those obtained with $\Gamma_{\alpha\beta}^\mu$ instead of $F_{\alpha\beta}^\mu$ ⁽⁵⁾. However, as we are working with $g^{\mu\nu}$, $F_{\alpha\beta}^\mu$ is more readily computed than $\Gamma_{\alpha\beta}^\mu$.

Let us now remember that

$$M = m + \underset{1}{m} + \underset{2}{m} + \dots; \quad \dot{v}^k = \dot{a}^k + \underset{1}{a}^k + \underset{2}{a}^k + \dots,$$

⁽³⁾ L. INFELD and J. PLEBAŃSKI: *Bull. Acad. Pol. Sc., Cl. III*, **4**, 689 (1956); **5**, 51 (1957).

⁽⁴⁾ L. INFELD: *Acta Phys. Pol.*, **16**, 177 (1957); *Rev. Mod. Phys.*, **29**, 398 (1957).

⁽⁵⁾ E. SCHRÖDINGER: *Space-Time Structure* (Cambridge, 1954), p. 55.

where \underline{m} and \underline{a}^k are small quantities of the $(n+2)$ -nd order. Let us define

$$\underline{b}^{\mu} = \underline{a}^{\mu} + \frac{\overline{F_{00}^{\mu}}}{n+2} + 2v^k \frac{\overline{F_{0k}^{\mu}}}{n+1} + v^k v^l \frac{\overline{F_{kl}^{\mu}}}{m}.$$

Equation (2) is then equivalent to

$$(3) \quad \dot{\underline{m}} + \sum_{p=0}^{p=n-1} \underline{m}_{n-p-1} \underline{b}^p = 0,$$

$$(4) \quad \dot{\underline{m}} v^k + \sum_{p=0}^{p=n} \underline{m}_{n-p} \underline{b}^p = 0.$$

The problem of motion has thus been reduced to the determination of the \underline{b}^{μ} .

2. – Newtonian approximation.

We obtain the Newtonian approximation by solving (3) and (4) for $n = 0$. From (3) we get $\dot{\underline{m}} = 0$, whence, from (4) $\underline{b}_0^k = 0$. It follows

$$\underline{a}^k = -\frac{\overline{F_{00}^k}}{2} = \frac{1}{4} \underline{g}_{2,k} = \frac{\partial}{\partial x^k} \sum \underline{m},$$

i.e., Newton's law of attraction.

3. – Post-Newtonian equations of motion.

For $n = 1$, (3) gives

$$\dot{\underline{m}}_1 = -\underline{m}_0 \underline{b}^0 = -\underline{m} \underline{F}_{200}^0 \equiv 0,$$

whence, we deduce from (4), $\underline{m} \underline{b}_1^k = 0$, i.e.

$$\underline{a}_1^k = -\underline{F}_{300}^k - 2v^l \underline{F}_{20l}^k \equiv 0.$$

There are no first-order corrections, either to mass or to acceleration. For $n = 2$, (3) gives

$$\dot{\underline{m}}_2 = -\underline{m} \underline{b}^0 = -\underline{m} (\overline{F_{00}^0} + 2v^k \overline{F_{0k}^0}) = \underline{m} \left(3 \frac{\partial}{\partial t} \sum \frac{\underline{m}}{R} + 4v^k \frac{\partial}{\partial x^k} \sum \frac{\underline{m}}{R} \right).$$

Now

$$\overline{\frac{\partial}{\partial t} \sum \frac{m}{R}} = \overline{\sum v^k \frac{\partial}{\partial \xi^k} \left(\frac{m}{R} \right)} = \frac{d}{dt} \overline{\sum \frac{m}{R}} - v^k \frac{\partial}{\partial x^k} \overline{\sum \frac{m}{R}}.$$

Thus

$$\frac{\dot{m}}{2} = m \left(v^k a^k + 3 \frac{d}{dt} \sum' \frac{m}{R} \right).$$

Whence

$$\frac{\dot{m}}{2} = \frac{1}{2} m v^2 + 3m \sum' \frac{m}{R}.$$

This result has been discussed in I.

We then obtain from (4) $\frac{\dot{m}}{2} v^k + m b^k = 0$, whence

$$\frac{a^k}{2} = - \left[\frac{\dot{m}}{m} v^k + \frac{F_{ln}^k}{2} v^l v^n + 2v^l \frac{F_{0l}^k}{3} + \frac{F_{00}^k}{4} \right],$$

where

$$\begin{aligned} F_{ln}^k &= -\frac{1}{4} \delta_{ln} \mathfrak{g}_{\frac{1}{2},k}^{00}; & F_{0l}^k &= \frac{1}{2} (\mathfrak{g}_{\frac{1}{3},k}^{0l} - \mathfrak{g}_{\frac{1}{3},l}^{0k}), \\ F_{00}^k &= \frac{1}{2} \mathfrak{g}_{\frac{1}{2},\frac{1}{2}}^{00} \mathfrak{g}_{\frac{1}{2},k}^{00} - \mathfrak{g}_{\frac{1}{3},0}^{0k} - \frac{1}{4} (\mathfrak{g}_{\frac{1}{4},0}^{00} + \mathfrak{g}_{\frac{1}{4},l}^{0l}). \end{aligned}$$

All the needed quantities have already been computed in I.

4. – The perihelion advance.

If we explicitly work out the previous formula for the case of a body of mass m and velocity v^k revolving around another body of mass M , taking for simplicity the center of mass at rest, we get, by a tedious but straightforward computation:

$$\begin{aligned} \frac{a^k}{2} = -\frac{MR^k}{R^3} \left[\left(1 + \frac{4m}{M} + \frac{2m^2}{M^2} \right) v^2 - \left(4 + \frac{5m}{M} \right) \frac{M}{R} - \frac{3m^2}{2M^2} \left(\frac{R^l v^l}{R} \right)^2 \right] + \\ + \left(4 + \frac{7m}{M} + \frac{3m^2}{M^2} \right) \frac{Mv^k}{R^2} \left(\frac{R^l v^l}{R} \right), \end{aligned}$$

where R^k is the vector from M to m .

If $m \ll M$, we have, in ordinary vector notation

$$\mathbf{a} = -\frac{MR}{R^3} \left(1 + \mathbf{v}^2 - 4 \frac{M}{R} \right) + \frac{4M\mathbf{v}}{R^2} \left(\frac{\mathbf{R} \cdot \mathbf{v}}{R} \right).$$

This equation can now be handley by the usual technique (6). In polar co-ordinates, it reads

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\frac{M}{r^2} \left(1 - 4\frac{M}{r} + \dot{r}^2 + r^2\dot{\theta}^2 \right) + 4M\frac{\dot{r}^2}{r^2}, \\ \frac{(r^2\dot{\theta})}{r} = 4M\frac{r\dot{\theta}}{r}\dot{r}. \end{cases}$$

This last equation is readily integrated

$$r^2\dot{\theta} = l \exp[-4M/r],$$

where l is a constant.

We pass to a new variable $u = 1/r$, and to θ (primed) derivatives, instead of time (dotted) derivatives (6) and get:

$$(5) \quad u' = -u + \frac{M}{l^2} + Mu^2 + Mu'^2 + \frac{M}{l^2} (\exp[8Mu] - 1 - 4Mu \exp[8Mu]).$$

Till now, there have been no approximations, except those explicitly stated.

We now suppose that $Mu \ll 1$, and replace u' , in the right-hand member, by its value deduced from $u = M(1 + e \cos \theta)/l^2$, which is the approximate solution of (5).

We obtain

$$u'' = -u + \frac{M}{l^2} \left[1 - \frac{M^2}{l^2} (1 - e^2) \right] + 6 \frac{M^2}{l^2} u,$$

which shows that the perihelion advance, per revolution, is

$$\frac{6\pi M^2}{l^2} = \frac{6\pi M}{a(1 - e^2)},$$

where a is the semi-major axis of the « ellipse », and e its excentricity. This result is in agreement with that computed by elementary methods (7), and with experimental data. This is a good test for our theory, as some other approximation methods lead to wrong results in this point (8,9).

(6) H. GOLDSTEIN: *Classical Mechanics* (Cambridge, Mass., 1951), Ch. 3.

(7) H. WEYL: *Space Time Matter* (English translation, New York, 1922), p. 258.

(8) F. J. BELINFANTE: *Phys. Rev.*, **89**, 914 (1953).

(9) P. HAVAS: *Phys. Rev.*, **108**, 1351 (1957).

5. – Radiation field.

We call *radiation field* those terms which depend on ε . As ε is an arbitrary constant, they can be treated quite independently of others. We shall denote by stars (*e.g.* $\overset{*}{g}^{\mu\nu}$) the quantities that are proportional to ε ; (quantities proportional to ε^2 appear only in the eighth order, and we shall not have to deal with them).

We have seen in I that $\overset{*}{g}^{\mu\nu}$ vanishes in the 3-rd order as a consequence of the law of conservation of mass, and in the 4-th order, because of the conservation of linear momentum. Radiation terms thus first appear only in the 5-th order of the metric:

$$\left\{ \begin{array}{l} \overset{*}{g}_{5}^{00} = \overset{*}{h}_{5}^{00} = 4\varepsilon \sum \left[\dot{m} + \frac{1}{3!} m(\ddot{R}^2) \right], \\ \overset{*}{g}_{5}^{0k} = \overset{*}{h}_{5}^{0k} = 0, \\ \overset{*}{g}_{5}^{kl} = \overset{*}{h}_{5}^{kl} = 4\varepsilon \sum m(v^k a^l + v^l a^k). \end{array} \right.$$

These expressions depend only on time, because, by Newton's law of attraction

$$\sum m(\ddot{R}^2) = 2 \sum m v^2 - \sum' \frac{\overset{A}{m} \overset{B}{m}}{D},$$

is a space-independent quantity, at this order of approximation.

It follows that $\overset{*}{g}^{\mu\nu}$ contributes to the curvature tensor $R_{\nu\sigma}^{\mu}$ only in the seventh order, and thus can be annihilated by an infinitesimal co-ordinate transformation (10, 11). The consequences of this fact will be examined in a later publication.

The next approximation is:

$$\left\{ \begin{array}{l} \overset{*}{g}_6^{00} = 0, \\ \overset{*}{g}_6^{0k} = 4\varepsilon \sum \left[(\dot{m} v^k) + m \dot{a}^k + \frac{m}{2} (\ddot{v}^k \ddot{R}^2) - \frac{1}{3!} (\ddot{m} \ddot{R}^k) \right], \\ \overset{*}{g}_6^{kl} = 0. \end{array} \right.$$

(10) L. INFELD and A. E. SCHEIDECKER: *Can. Journ. Math.*, **3**, 195 (1951).

(11) A. E. SCHEIDECKER: *Phys. Rev.*, **82**, 883 (1951); *Rev. Mod. Phys.*, **25**, 451 (1953).

(The origin of the last term of $\overset{*}{g}_{\frac{6}{6}}^{0k}$ is explained in I.) We see that $\overset{*}{g}_{\frac{6}{6}}^{0k}$ is space-dependent, but

$$\overset{*}{g}_{\frac{6}{6}}^{0k},_l - \overset{*}{g}_{\frac{6}{6}}^{0l},_k = \frac{4}{3}\varepsilon \sum m(v^k R^l - v^l R^k) = 0 ,$$

by the law of conservation of angular momentum, which holds in this approximation, as a consequence of Newton's law of motion. It follows that $\overset{*}{g}_{\frac{6}{6}}^{0k}$ also contributes to $R_{\nu\sigma}^\mu$ only in the seventh order, and therefore can be annihilated.

Moreover, as the laws of motion can be written in an invariant form with the help of $R_{\nu\sigma}^\mu$ (12,13), it also follows that radiative effects can appear only at the seventh order (*) (14).

Indeed, it can be directly verified, with the help of the equations of motion (3) and (4), that

$$\overset{*}{m}_3 = \overset{*}{m}_4 = \overset{*}{a}_3^k = \overset{*}{a}_4^k = 0 .$$

Further, we have

$$\left\{ \begin{array}{l} \overset{*}{h}_{\frac{7}{7}}^{00} = 4 \sum \frac{\overset{*}{m}}{R} + \varepsilon \overset{*}{m} + \varepsilon \frac{(m \overset{...}{R^2})}{3!} + \varepsilon \frac{m}{5!} (\overset{....}{R^4}) + \left[\begin{array}{c} \text{corrective terms} \\ \text{from } \overset{*}{g}_{\frac{5}{5}}^{00} \end{array} \right], \\ \overset{*}{h}_{\frac{7}{7}}^{0k} = 0 , \\ \overset{*}{h}_{\frac{7}{7}}^{k\ell} = 4\varepsilon \sum m(v^k a^\ell + v^\ell a^k) + (m v^k v^\ell) + \frac{m}{3!} (v^k v^\ell \overset{...}{R^2}), \end{array} \right.$$

where the « corrective terms from $\overset{*}{g}_{\frac{5}{5}}^{00}$ » result from $\dot{v}^k = a^k + \overset{*}{a}_{\frac{2}{2}}^k$.

We must add to this

$$\overset{*}{g}_{\frac{7}{7}}^{00} = 2\overset{*}{g}_{\frac{5}{5}}^{00} \sum \frac{m}{R} - 2\overset{*}{g}_{\frac{5}{5}}^{k\ell} \sum \frac{m R^k R^\ell}{R^3} ,$$

which results from the integration of

$$\nabla_{\frac{7}{7}}^2 \overset{*}{g}_{\frac{7}{7}}^{00} = \overset{*}{\Theta}_{\frac{7}{7}}^{00} = \overset{*}{g}_{\frac{5}{5}}^{k\ell} \overset{*}{g}_{\frac{2}{2}}^{00} ,$$

(12) J. L. SYNGE and A. SCHILD: *Tensor Calculus* (Toronto, 1949), p. 93.

(13) F. A. E. PIRANI: *Acta Phys. Pol.*, **15**, 389 (1956); *Bull. Acad. Pol. Sci., Cl. III*, **5**, 143 (1957); *Phys. Rev.*, **105**, 1089 (1957).

(*) I.e., as fifth-order corrections to masses and accelerations. This corresponds to Infeld's ninth-order equation of motion.

(14) L. INFELD: *Phys. Rev.*, **53**, 836 (1938).

and

$$\frac{\dot{g}^{kl}}{7} = 4\varepsilon \sum \left\{ -\frac{1}{4} [\overset{m}{2}(v^k R^i + v^i R^k)] - \frac{m}{4} (\overset{a^k}{2} R^i + \overset{a^i}{2} R^k) + \frac{1}{10} (\overset{m}{2} \ddot{R}^k R^i) - \frac{\delta^{kl}}{30} (\overset{m}{2} \ddot{R}^2) \right\},$$

which is needed in order to fulfil de Donder's condition $\overset{*}{g}_{6,0}^{k0} + \overset{*}{g}_{7,l}^{kl} = 0$. Note that $\nabla^2 \overset{*}{g}^{kl} = 0$.

6. - Radiation reaction.

From (3), we get

$$\overset{\dot{m}}{5} = -mb_4^0 = -m\overset{*}{F}_{00}^0 = \frac{m}{4} (3\overset{*}{g}_{5,0}^{00} - \overset{*}{g}_{5,0}^{ll})_{,0}.$$

It follows

$$\overset{\dot{m}}{5} = \varepsilon m \frac{d}{dt} (3T - 17V),$$

where

$$T = \frac{1}{2} \sum mv^2 \quad \text{and} \quad V = -\frac{1}{2} \sum' \frac{\overset{A}{m} \overset{B}{m}}{D}.$$

(Note that $T+V$ is a constant of the motion, at this approximation). Next, we have, from (4)

$$\overset{\dot{a}}{5}^k = -\frac{\overset{\dot{m}}{5}}{m} v^k - 2v^l \overset{*}{F}_{0l}^k - \overset{*}{F}_{00}^k,$$

where

$$\overset{*}{F}_{0l}^k = \frac{1}{2} \overset{*}{g}_{5,0}^{kl} - \frac{1}{2} (\overset{*}{g}_{6,0}^{0k} - \overset{*}{g}_{6,0}^{0l}) = \frac{1}{2} \overset{*}{g}_{5,0}^{kl},$$

and

$$\overset{*}{F}_{00}^k = -\frac{1}{4} (\overset{*}{g}_{7,0}^{00} + \overset{*}{g}_{7,0}^{ll})_{,k} + \frac{1}{4} \overset{*}{g}_{5,0}^{km} \overset{*}{g}_{2,0}^{00,m} + \frac{1}{2} \overset{*}{g}_{5,0}^{00} \overset{*}{g}_{2,0}^{00,k} - \overset{*}{g}_{5,0}^{0k}.$$

All the needed quantities have already been computed.

7. - Radiated energy.

We define the radiated energy, per unit time, as minus the rate of work of radiative forces

$$(6) \quad U = - \sum mv^k \overset{\dot{a}}{5}^k.$$

We can also compute it with the help of the conservation law (15)

$$[g(\mathfrak{T}^{\mu\nu} + t^{\mu\nu})]_{\nu} = 0 .$$

Here

$$gt^{\mu\nu} = - \frac{1}{16\pi} (Q^{\mu\nu} + g^{\mu\gamma,\beta} g^{\nu\beta,\alpha}) ,$$

where use has been made of de Donder's condition, and the explicit form of $Q^{\mu\nu}$ is given in I. Thus

$$U = - \frac{\partial}{\partial t} \iiint g(\mathfrak{T}^{00} + t^{00}) dV = + \iint g t^{0k} d\sigma_k .$$

Such a computation has already been proposed by LANDAU and LIFSHITZ (16) on the basis of the « linear approximation » of Einstein's equations, that is, by keeping in gt^{0k} only terms which are quadratic in $\mathfrak{h}^{\mu\nu}_{,\alpha\beta}$. This method will be correct only if $\mathfrak{s}^{\mu\nu}$ falls faster than $\mathfrak{h}^{\mu\nu}$, at spatial infinity.

That this is *not* the case can be seen in the following way. Let us suppose that at spatial infinity $g^{\mu\nu} \approx \mathfrak{h}^{\mu\nu}$ falls like R^{-1} . The same will be true of $\mathfrak{h}^{\mu\nu}_{,\alpha\beta}$ (unless the sources are time-independent) and thus $\Theta^{\mu\nu}$ will fall like R^{-2} . It then follows from

$$\nabla^2 \mathfrak{s}^{\mu\nu} - \ddot{\mathfrak{s}}^{\mu\nu} = \Theta^{\mu\nu} ,$$

that $\mathfrak{s}^{\mu\nu}$ will not vanish for large R , but rather behave like $\log R$ unless $\Theta^{\mu\nu}$ has a null time-average. This shows that the whole problem of boundary conditions in General Relativity should be reconsidered. At any rate, this argument may cast serious doubts on the validity of the linear approximation.

8. – Two-body circular motion.

We shall explicitly evaluate U in the case of two bodies of masses m and M , revolving on circular orbits with constant angular velocity $\omega = \sqrt{(m+M)/D^3}$, D being the distance between them. As the final result may look paradoxical, and the computations are rather short, we shall quote them with some detail.

First we note that as the kinetic and potential energies are constant (at the needed approximation), \dot{m}_2 , \dot{m}_5 , \dot{g}^{00}_5 and \dot{g}^{11}_5 vanish. Next we choose, for

(15) L. LANDAU and E. LIFSHITZ: *The Classical Theory of Fields* (English translation; Cambridge, Mass., 1951), p. 317.

(16) L. LANDAU and E. LIFSHITZ: loc. cit., p. 330.

convenience, a system of co-ordinates where the center of mass is at rest:

$$mv^k + MV^k = 0.$$

As a consequence, we can neglect, in $\overset{*}{F}_{00}^k$, all spatial constants which give equal contributions to $\overset{*}{a}^k$ and $\overset{*}{A}^k$, as they will cancel in the computation of U .

Thus the only relevant term of $\overset{*}{h}^{60}$ is $4\varepsilon \sum_7 m(R^4)/5!$ and $\overset{*}{g}^{kl}$ gives no contribution.

Moreover

$$-\frac{1}{4} \overset{*}{g}_{,k}^{00} = \frac{1}{2} \overset{*}{g}_{,ln}^{ln} \sum \left(\frac{mR^l R^n}{R^3} \right)_{,k} = -\frac{3}{2} \overset{*}{g}_{,ln}^{ln} \sum \frac{mR^k R^l R^n}{R^5} + \overset{*}{g}_{,kl}^{kl} \sum \frac{mR^l}{R^3}.$$

The first term vanishes, as a result of the motion being circular, because $D'D^n(a^l v^n + a^n v^l) = 0$. The second term of $\overset{*}{g}_{,k}^{00}$ exactly cancels the third term of $\overset{*}{F}_{00}^k$. It remains still $-\overset{*}{g}_{,0}^{0k}$, of which the only relevant term is $-4\varepsilon \sum m(v^k R^2)/3!$

Thus

$$\overset{*}{a}^k = -v^l \overset{*}{g}_{,0}^{kl} + \varepsilon \sum_5 m \left[\frac{2}{3} (v^k R^2) + \frac{1}{30} (R^2 R^k) \right].$$

A straightforward computation, making use of $a^k = -MD^k/D^3$ and omitting all spacial constants of \sum , gives

$$\overset{*}{a}^k = \frac{\varepsilon}{30} m(m+M) \frac{v^k}{D^4} (221M - 269m),$$

whence

$$U = \varepsilon \frac{8}{5} \frac{m^2 M^2 (m+M)}{D^5}.$$

For outgoing waves (retarded potential, $\varepsilon = -1$), we obtain that the radiated energy is negative, *i.e.*, the mechanical system gains energy (D will slowly increase). This result is in qualitative agreement with that found by Hu⁽¹⁷⁾.

9. - Four-body circular motion.

If we consider four equal masses placed at the corners of a uniformly rotating square, we can expect that there are no radiative effects at this order, because the quadrupole moments are constant. This can indeed be easily

⁽¹⁷⁾ N. HU: *Proc. Roy. Irish Acad.*, A 51, 87 (1947).

verified by noting that $\overset{\ast}{g}_{\frac{5}{5}}^{kl} = 0$, and therefore $\overset{\ast}{a}_{\frac{5}{5}}^k$ reduces to

$$\overset{\ast}{a}_{\frac{5}{5}}^k = \varepsilon \sum m \left[\frac{2}{3} (\overset{\dots}{v^k R^2}) + \frac{1}{30} (\overset{\dots}{R^2 R^k}) \right].$$

The explicit evaluation of this quantity shows that the contribution of the farthest particle is exactly cancelled by that of the two nearer ones, *i.e.*, $U = 0$.

It may be concluded that the hypothesis that gravitational waves are, at the lowest approximation, quadrupole ones, is a reasonable assumption. If a general proof could be found, the radiated energy would be, in ordinary units (16):

$$U = - \frac{G}{180c^5} (\overset{\dots}{D^{kl}} \overset{\dots}{D^{kl}}),$$

where

$$D^{kl} = \sum m(3\xi^k \xi^l - \delta^{kl} \xi^n \xi^n).$$

This result is extremely striking, because it is exactly the same as for *electromagnetic quadrupole radiation* (18), if we merely replace m by e and $(-G)$ by $(1/4\pi\varepsilon_0)$, *i.e.*, if we replace Newton's law by Coulomb's law!

10. – Negative energy.

We still have to discuss the «unbelievable result» (11) that, as a consequence of gravitational radiation, our system gains energy. This implies that gravitational waves (generated by a system of freely gravitating pole particles) carry away a *negative* flux of energy. As a matter of fact, this result becomes quite reasonable if we recall that the energy density of the gravitational field

$$\overset{\ast}{g}_{\frac{4}{4}}^{00} = - \frac{1}{16\pi} Q_{\frac{4}{4}}^{00} = - \frac{7}{128\pi} \overset{\ast}{g}_{\frac{2}{2}}^{00,k} \overset{\ast}{g}_{\frac{2}{2}}^{00,k},$$

is also negative. We can write it, in ordinary units

$$\overset{\ast}{g}_{\frac{4}{4}}^{00} = - \frac{7}{8\pi G} \mathbf{g}^2,$$

where \mathbf{g} is the gravitational field. Apart from the factor 7, this is the exact equivalent of the electrostatic field energy density $+(\varepsilon_0/2)\mathbf{E}^2$. If we integrate

(18) L. LANDAU and E. LIFSHITZ: loc. cit., p. 206.

the latter throughout the whole space, it is equal to the total potential energy⁽¹⁹⁾. In a similar fashion $\int g t^{00} = 7V$. If we add this result to $g \mathfrak{T}^{00}$, computed up to the fourth order, included, we find

$$\int g(\mathfrak{T}^{00} + t^{00}) = (\sum m) + (T - 6V) + 7V = (\sum m) + T + V,$$

i.e., the total (rest + kinetic + Newtonian potential) energy.

The electrostatic analogy still holds, although it is less trivial than in the previous section, because rest-mass is not conserved, while charge is.

* * *

I am indebted to Professor N. ROSEN for having suggested this problem to me, and for stimulating discussions.

⁽¹⁹⁾ W. PANOFSKY and M. PHILLIPS: *Classical Electricity and Magnetism* (Cambridge, Mass., 1955), p. 88.

RIASSUNTO (*)

Dalle successive soluzioni approssimate delle equazioni di Einstein vincolate alle condizioni di de Donder si deducono leggi di moto per particelle lente secondo la relatività generale. Le prime tre approssimazioni danno risultati identici a quelli del precedente lavoro⁽¹⁾ (equazioni di moto newtoniane e postnewtoniane). Nelle approssimazioni superiori compaiono contributi dal campo radiativo. Si trova che questi possono trattarsi indipendentemente da altri effetti relativistici e producono correzioni delle masse a riposo e delle accelerazioni delle particelle. In particolare si dimostra che in conseguenza di radiazione gravitazionale un sistema di particelle *acquista* energia, sottintendendo così che le onde gravitazionali asportino una *quantità negativa* di energia. Il che si rende plausibile dimostrando che anche la densità d'energia del campo gravitazionale è una grandezza negativa. Si propone per questi risultati un'analogia elettromagnetica.

(*) Traduzione a cura della Redazione.

**The Construction and Working
of a Wide Line Nuclear Magnetic Resonance Spectrometer
and the Measurements of Some Chemical Shifts.**

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(ricevuto il 10 Novembre 1958)

Summary. — A wide line nuclear magnetic resonance spectrometer of high sensitivity operating between 4 and 10 MHz, has been described together with a high stability electromagnet system. Using the spectrometer, the chemical shifts of ^{17}O resonance in several compounds have been measured. The results have been interpreted in terms of CO bond character in various groups of organic compounds.

1. – Introduction.

The nuclear magnetic resonance spectroscopy is based on the fact that the various isotopes of the elements can be distinctly identified because of their differing nuclear gyromagnetic ratios. Each isotope, with non-zero spin, possesses a magnetic moment, and behaves as though it were a tiny magnet with a unique strength. So, if matter be placed in a steady magnetic field H_0 , say, in the Z-direction, the nuclear magnetic moments will tend to orient themselves parallel to the field. After the establishment of thermal equilibrium, there will be a net paramagnetic polarization in the Z-direction given by

$$(1) \quad M_0 = \chi H_0,$$

where χ is the nuclear magnetic susceptibility. If now, we superimpose an oscillating magnetic field in the x -direction such that

$$(2) \quad H_x = 2H_1 \cos \omega t,$$

the polarization vector starts deviating from the Z -direction and reaches the maximum deviation when the Z -field H_0 approaches the resonance value H given by Larmor's relation

$$(3) \quad |\gamma|H = \omega,$$

where γ is the gyromagnetic ratio of the nucleus under consideration and ω is the precessional frequency. It is well known that the resonance phenomenon was simultaneously discovered in 1946 by BLOCH and his collaborators at Stanford ⁽¹⁾ using the «crossed-coil» method, and PURCELL and his co-workers at Harvard ⁽²⁾ employing the bridge method. We have essentially followed the technique developed by BLOCH and our spectrometer is similar in principle to those of PROCTOR ⁽³⁾ and WEAVER ⁽⁴⁾.

2. – Description of the spectrometer.

The spectrometer consists of an R.F. transmitter, a probe, an R.F. pre-amplifier, an R.F. amplifier, a detector, an audio frequency amplifier and an oscilloscope. To record very weak signals, a narrow band amplifier with a phase-sensitive detector and a graphic recorder have also been included. The steady field of several kilogauss is produced by an electromagnet with its associated power supply and current regulator.

The transmitter ⁽⁴⁾ consists of an R.F. oscillator operating between 4 and 10 MHz and a power amplifier which produces enough current for an adequate R.F. field in the transmitter coil. A 6SN7 double triode serves as an R.F. oscillator which drives enough current at the grids of the 832A power amplifier. With -45 V as grid bias and 300 V on the plates of the 832A tube, by varying the screen-grid voltage from -10 V to $+30$ V, one can get R.F. fields in the transmitter coil ranging from 100 milligauss to 2 gauss. The power amplifier serves to isolate the transmitter coil from the load variations as also to supply adequate and variable R.F. current to the coil. A twin lead co-axial line feeds the R.F. power to the probe which is kept inside the gap of the polefaces of the electromagnet. As the transmitter and the receiver are located close to each other, adequate precaution is taken to minimize the

⁽¹⁾ F. BLOCH: *Phys. Rev.*, **70**, 460 (1946).

⁽²⁾ E. M. PURCELL, H. C. TORREY and R. V. POUND: *Phys. Rev.*, **69**, 37 (1946).

⁽³⁾ W. G. PROCTOR: *Phys. Rev.*, **79**, 35 (1950).

⁽⁴⁾ H. E. WEAVER: *Phys. Rev.*, **89**, 923 (1953).

unwanted direct coupling between the two units which will result in an excessive leakage voltage at the detector. The transmitter is mounted in a rectangular brass box which is fixed rigidly to a platform to minimize microphonics. The regulated power supplies, which serve the two units, are adequately decoupled to reduce interference between them.

It is needless to say that the probe or the « Head » is the most delicate and sensitive part of the spectrometer. The construction of the « Head » is similar to that of Weaver's (⁴). It contains two coils mounted orthogonally and rigidly on a perspex frame. The transmitter coil is of the Helmholtz type and one half of the coil can be moved relative to the other coil known as receiver coil. The receiver coil holds the test tube containing the sample and is electrically shielded by a Faraday cage from the transmitter coil. In addition to the two usual controls known as *U* and *V* mode controls which effectively decouple the two coils and introduce a leakage of the desired mode at the detector, there is also an additional arrangement by which the receiver coil can be moved vertically up and down relative to the transmitter coil. The reasons for such an elaborate system of controls are apparent from the following considerations.

For an R.F. field of the order of 0.5 G, about 50 V exist across the transmitter coil. If the two coils can be mounted exactly perpendicular to each other, then there will be no induced voltage in the receiver coil. But there is always a practical limit to which this accuracy can be achieved and usually the tuned receiver coil may pick up voltage as high as 5 V. But, the maximum allowable voltage across this coil is of the order of a few millivolts, as otherwise, this voltage, after amplification, will saturate the other stages of the R.F. amplifier, resulting in the loss of the efficiency of the spectrometer. By moving either the transmitter or the receiver coil relative to each other, with the help of a differential screw arrangement, considerable decoupling can be achieved. The remaining voltage can be removed by the operation of the *U* and *V* mode paddles. The *V* mode paddle (⁵) consists of a semicircular copper disc of about 0.5 mm thickness while the *U* mode (⁴) control consists of a resistance and an inductance in series in the form of a loop. By a judicious operation of the two controls, complete decoupling can be effected, and any desired mode of leakage can be introduced. This leakage serves to make the 1N34 crystal diode detector work efficiently and also determines the shape of the nuclear induction signal. The whole assembly is mounted rigidly in a brass box on whose outer covers are pasted two ten-turn coils of Helmholtz type which are supplied with an audio-frequency current to modulate the nuclear signal.

(⁵) F. BLOCH, W. W. HANSEN and M. E. PACKARD: *Phys. Rev.*, **70**, 474 (1946).

The receiver, which consists of an R.F. preamplifier, R.F. amplifier, a detector and an A.F. amplifier, is shown in Fig. 1. The detected nuclear signal is enhanced by a factor of Q (about 80 in our case) of the receiver coil using a tuned circuit. This voltage is further amplified by a low noise amplifier employing Wallman's circuit which has a gain of about six. The amplified signal is fed into a six stage R.F. broadband amplifier having a total gain of fifty. The modulated R.F. signal is finally detected with an 1N34 crystal diode.

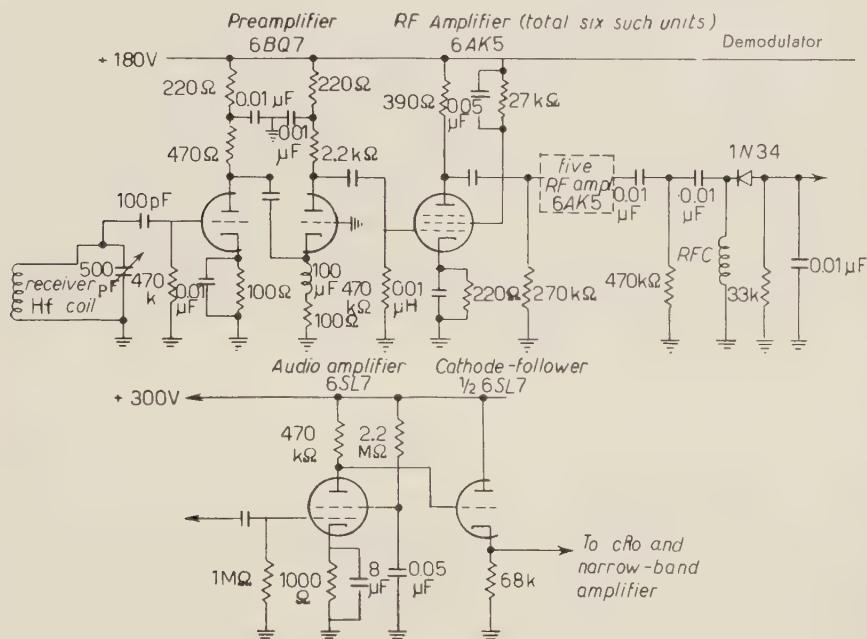


Fig. 1. — Receiver.

The detected audio signal is further amplified and applied to the Y plates of a 304A DuMont oscilloscope. The X-axis plates are supplied with a voltage derived from the modulating coils.

Strong signals, which have a large signal to noise ratio, can be displayed thus on the oscilloscope. However, many signals cannot be displayed on a scope as they will be considerably less than the noise of the system itself. Such signals can still be observed by using a « lock-in-amplifier » with a twin-T-filter which rejects all other than the fundamental frequency of modulation. We have adopted a circuit similar to the one used by BLOEMBERGEN (6) as

(6) N. BLOEMBERGEN: *Nuclear Magnetic Relaxation* (The Hague, 1948).

shown in the diagram of Fig. 2. The component of the nuclear signal is amplified in the first two stages and is applied to the grids of the two pentodes, used as a mixer with a mixing voltage of proper phase and the same modu-

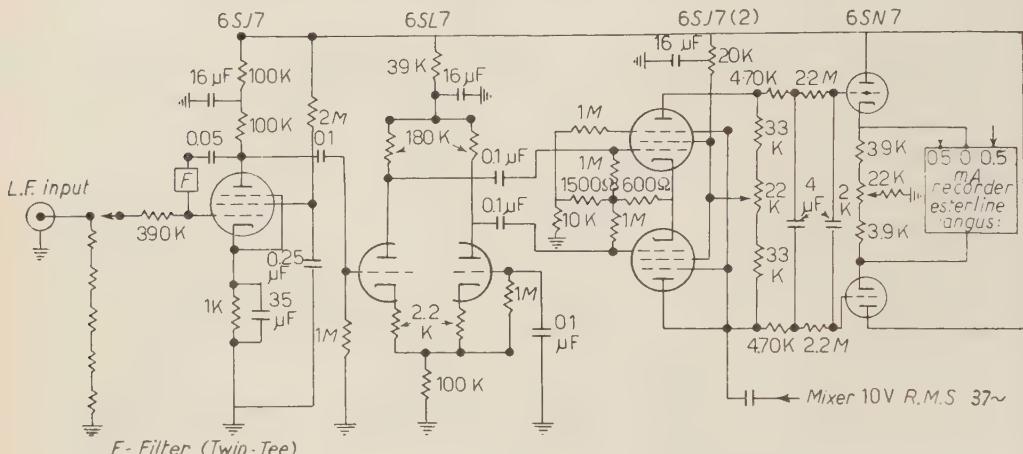


Fig. 2. — Narrow band amplifier with phase sensitive detector.

lation frequency. This eliminates all spurious signals and noises. The detected voltage is amplified and smoothed by an R.C. filter. Then it is applied to the grids of the twin triode and the signal is recorded in the D.C. milliammeter which is connected across the cathodes of the twin triode instead of across the plates. We found that this modification resulted in a great increase in the efficiency in detecting weak signals. The magnitude of the current is proportional to the amplitude of the nuclear signal and its sign depends upon the phase of the input voltage. By utilizing this method within practical limits a large improvement in the signal to noise ratio (between four and five in the case of a ${}^2\text{D}$ signal in 0.5 m MnSO_4 solution) can be realized.

3. — The electromagnet system.

The electromagnet (*) used with the spectrometer has pole pieces about 9.5 in. in diameter and an air gap of 1.75 in. For a typical sample volume the homogeneity is better than $1 \text{ in } 10^5$ at the most favourable spots. In Figs. 3 and 4 we have shown the signals of ${}^2\text{D}$ in a 40% D_2O sample and of

(*) We are greatly indebted to Prof. HANS H. STAUB of Physikalisches Institut der Universität, Zürich, for providing us with an electromagnet constructed by Wild Muri Co., Switzerland, under his personal supervision.

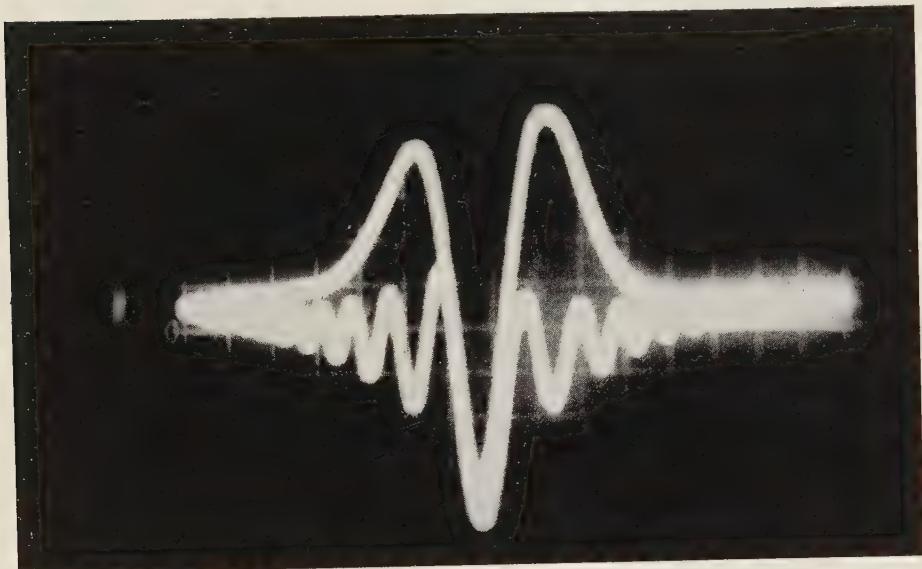


Fig. 3. - ${}^2\text{D}$ signal in 40% D_2O sample.

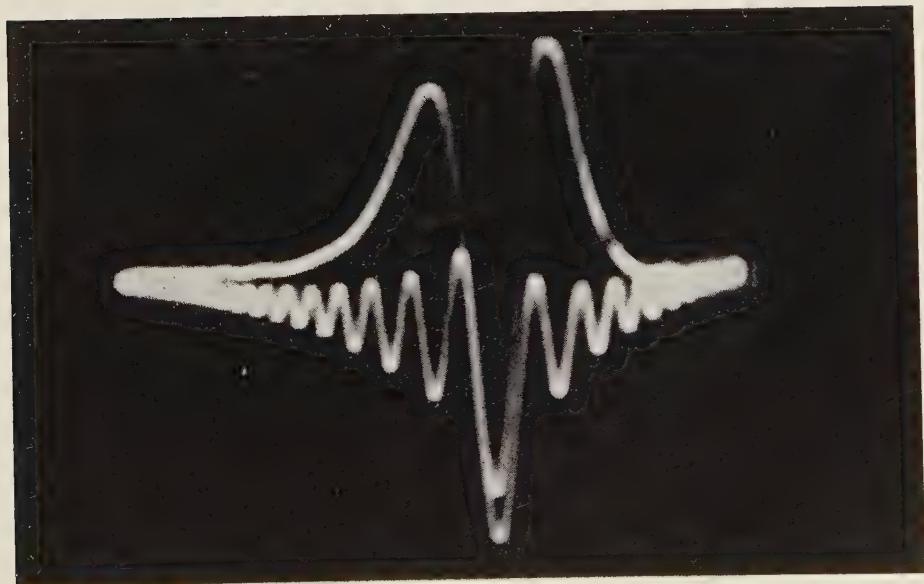


Fig. 4. - ${}^{23}\text{Na}$ signal in saturated NaOH solution.

^{23}Na in saturated NaOH solution with the wiggles taken at the fields of 8 kG and 5 kG respectively. There are two water cooled windings around each pole piece. Each section consists of about 20 000 turns and the magnetization arises from a large number of turns rather than from a large current. This feature makes it possible to regulate the current by hard tubes, specifically six 304 TL in parallel. The diagram showing the rectifier, the current regulator and the D.C. amplifier is given in Fig. 5. A maximum field of 15 kG

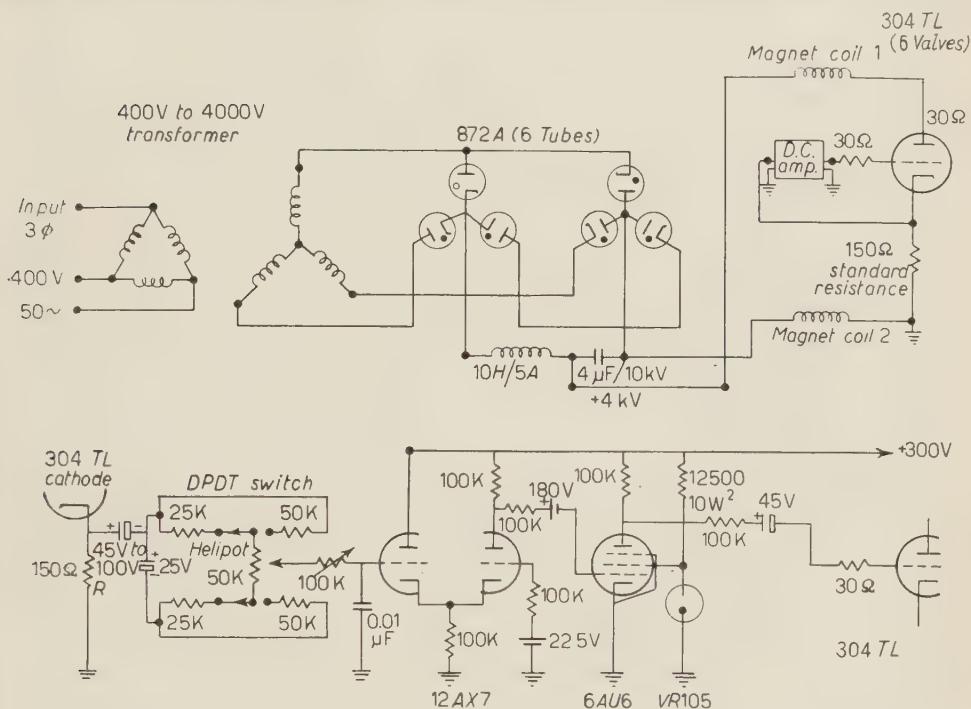


Fig. 5. — 4 kV magnet power supply and current regulator.

is obtained for a current of 2 A at 4 000 V. This voltage is derived by a step-up transformer by full wave bridge rectification from 400 V 3 phase lines using six 872 mercury rectifiers in parallel. To obtain current regulation, the magnet current is passed through a fixed resistor of high wattage which is immersed in an oil container. The difference in voltage across the resistor and that of a standard battery is amplified by a D.C. amplifier of two stages with a total gain of 2 000. The output is applied to the grids of 304 TL tubes, thus providing a feed back capable of reducing the line voltage variations of about 10% to current variations of 0.001% approximately. Thus an overall stability of about 1 in 10^5 is achieved in a typical operating range of 3 kG to 11 kG. The magnetic field, after three hours of warming up, does

not drift by more than 500 mG in an hour. The steady field is modulated by passing an audio frequency current of 37 Hz to the coils cemented on the plates of the nuclear induction head. This current is obtained using a Hewlett Packard 206A model audio oscillator with a power amplifier, similar to the one used by BLOEMBERGEN (⁶). For a current of 0.5 A in the sweeping coils, the modulating field is of the order of 5 G which facilitates the presentation of the signals on the oscilloscope. It is possible to vary the sweep field from 5 G to 20 mG. The mixer voltage, which is derived from the unit, is applied to the phase sensitive detector of the narrow band amplifier.

The power supplies for all the units of the spectrometer are well regulated and have a ripple content less than one millivolt r.m.s. The D.C. filament supply is obtained from a bridge rectifier. Though it has a ripple of 10 mV r.m.s., it does not contribute much to any pickup. This unit is more convenient than the batteries which require constant attention.

4. – The working of the spectrometer.

Usually liquid samples are preferred because they often lead to narrow lines, and the relaxation times can be controlled by the addition of paramagnetic salts like manganese sulphate or ferric nitrate. To obtain suitable conditions for the signal observation, the magnitudes of the sweep and the R.F. fields must be chosen carefully. Usually the R.F. field is kept about 0.5 G ($2 H_1$) and the sweep field is varied from 5 G to 0.1 G depending on whether the signal is to be observed on the oscilloscope or recorded on the charts of the D.C. milliammeter. The total receiver gain is normally kept around 100 and the leakage at the detector is about 0.2 V for the efficient operation of the diode. The twin-T-filter in the narrow band amplifier has a band width of about 1 Hz and we have used total output time constants of 10, 20, 40 and 60 s. The frequency of the R.F. oscillator can be varied from 4 to 10 MHz and the frequency is measured very accurately using a Hewlett Packard frequency meter. The magnetic field is changed with the help of a 50 k Ω helipot either by hand for a quick change or by a motor at the rate of about 1 G per minute through a range of 2 000 G for a slow and steady change. A somewhat lower rate of field scanning can be achieved by employing a step down gear arrangement for the motor drive.

5. – The efficiency of the spectrometer.

There are two nuclei whose signals serve as severe tests of the optimum efficiency of the spectrometer. They are ^2D and ^{17}O which occur in nature in such low abundances as 150 and 350 parts per million respectively in or-

dinary water. Both the nuclei possess small quadrupole moments, which due to interaction with the molecular fields, should attain favourable relaxation times. However, we found it necessary to prepare a 0.5 molar $MnSO_4$ solution to record a good signal of 2D in 3 cm^3 of water at 8.5 kG. For the case of the

^{17}O nucleus, we used ordinary water without any paramagnetic catalyst and obtained a signal of ^{17}O with signal to noise ratio of about 20 in a field of 9 kG. The two signals

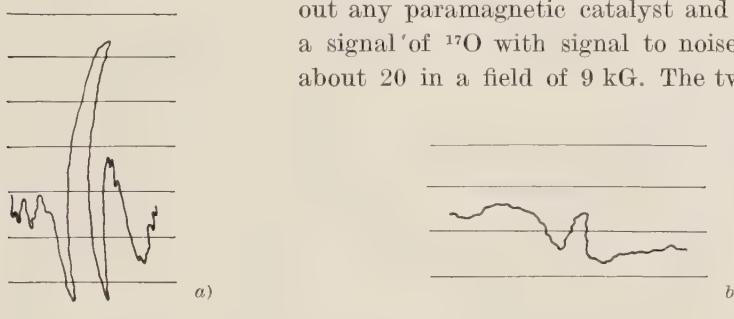


Fig. 6. - a) The derivative of u -mode signal of 2D in water containing 0.5 M $MnSO_4$ solution. R.F. field of 0.3 G (H_1) and sweep field of 0.4 G (H_s) are used. b) 2D signal with audio gain reduced to $\frac{1}{4}$ of that used in the recording of Fig. 6a. The other parameters are the same.

are shown in Figs. 6 and 7. An equally strong signal of ^{17}O was recorded using concentrated nitric acid as sample. The absorption mode signal of ^{17}O in concentrated HNO_3 is shown in Fig. 8.

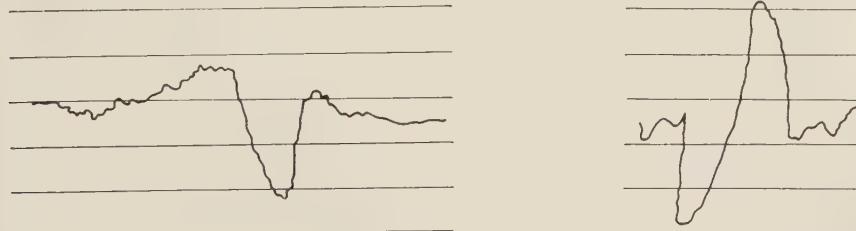


Fig. 7. - ^{17}O signal, the derivative of v -mode, obtained in 3 cm^3 of ordinary water. R.F. field of about 0.25 G (H_s) are used. Audio gain is about $\frac{1}{3}$ of that used in the tracing of Fig. 6a.

To test the overall efficiency of the spectrometer, we studied the temperature and the field dependence of ^{59}Co resonance shifts and compared the results with those obtained by another group in our laboratory using the

Fig. 8. - ^{17}O signal, the derivative of v -mode, obtained using concentrated HNO_3 as sample. The other parameters are the same as those used in the recording of Fig. 7.

Varian spectrometer (?). PROCTOR and YU (§) studied the resonance of ^{59}Co in several cobalt complexes and found the chemical shifts as large as 1.3% between two compounds such as potassium cobaltioxalate and potassium cobalticyanide. It was explained by RAMSEY (¶) that there exist several excited electronic energy levels close to the ground state and that there should therefore be a slight temperature dependence of the chemical shift in ^{59}Co resonance. We studied four compounds, *viz.*, $\text{CO}(\text{NH}_3)_6\text{Cl}_3$, $[\text{Co}(\text{NH}_3)_5\text{CO}_3]\text{NO}_3$, $\text{Na}_3\text{CO}(\text{NO}_2)_6$ and $[\text{Co}(\text{NH}_3)_5\text{Cl}]\text{Cl}_2$ at a resonance field of 5.7 kG for the temperature dependence of the chemical shifts. These samples were heated to 85 °C and the signals were observed on the calibrated screen of the oscilloscope at this temperature. The resonance field of each sample at room temperature (27 °C) was observed immediately on the oscilloscope screen by removing the hot sample and replacing it with the ordinary sample. The difference in field for each sample is given below:

TABLE I.

Sample	Difference in field ($H_{27\text{ }^\circ\text{C}} - H_{85\text{ }^\circ\text{C}}$)
$\text{Na}_3\text{CO}(\text{NO}_2)_6$	+ 0.8 gauss
$[\text{Co}(\text{NH}_3)_5\text{CO}_3]\text{NO}_3$	+ 0.7 »
$[\text{Co}(\text{NH}_3)_5\text{Cl}]\text{Cl}_2$	+ 0.75 »
$\text{Co}(\text{NH}_3)_6\text{Cl}_3$ (*)	+ 0.7 »

(*) PROCTOR and YU (§) reported a shift in frequency of about 1 kHz for the same difference in temperature at the resonance frequency of about 7000 kHz for $\text{Co}(\text{NH}_3)_6\text{Cl}_3$ solution.

As the difference in the field for each sample was observed in the calibrated oscilloscope, the error is of the order of 100 mG. Within this experimental error, our results are in good agreement with those of PROCTOR and YU (§).

The field dependence of the resonance was checked in a few compounds at different fields such as 5.7 kG and 7.8 kG. The measurements at the latter field were done using the Varian V4200B spectrometer. The results of the two experiments are given in terms of the dimensionless quantity σ defined in Table II. They are in excellent agreement showing thereby an extremely satisfactory working and the high fidelity of our spectrometer.

(?) S. S. DHARMATTI, C. R. KANEKAR and S. C. MATHUR: *Proceedings of International Conference on the Peaceful Uses of Atomic Energy*, 1958 (to be published).

(§) W. G. PROCTOR and F. C. YU: *Phys. Rev.*, **81**, 20 (1951).

(¶) N. F. RAMSEY: *Phys. Rev.*, **86**, 243 (1952).

TABLE II. - *Chemical shifts of ^{59}Co in cobalt compounds.*

No.	Compound	Shift in G at 5700 G 1	Shift in G at 7800 G 2	$\sigma \times 10^4$		Remarks
				1	2	
1	$[\text{Co}(\text{NH}_3)_4(\text{NO}_2)_2]\text{NO}_3$	+ 6.8	+ 9.35	+ 11.9	+ 12.0	Cis-isomer
2	$[\text{Co}(\text{NH}_3)_4(\text{NO}_2)_2]\text{NO}_3$	+ 5.20	+ 7.25	+ 9.1	+ 9.3	Trans-isomer
3	$[\text{Co}(\text{NH}_3)_5(\text{NO}_2)]\text{Cl}_2$	+ 3.5	+ 4.8	+ 6.1	+ 6.2	Yellow solution
4	$[\text{Co}(\text{NH}_3)_5\text{ONO}]\text{Cl}_2$	+ 3.5	+ 4.8	+ 6.1	+ 6.3	Orange solution
5	$\text{Na}_3\text{Co}(\text{NO}_2)_6$	+ 0.35 + 4.15	— + 5.7	+ 0.6 + 7.3	— + 7.3	Weak signal Strong signal
6	$\text{Co}(\text{NH}_3)_6\text{Cl}_3$	0	0	0	0	Reference sample
7	$[\text{Co}(\text{NH}_3)_5\text{CO}_3]\text{NO}_3$	— 5.2	— 7.2	— 9.1	— 9.2	
8	$[\text{Co}(\text{NH}_3)_5\text{Cl}]\text{Cl}_2$	— 5.6	— 7.7	— 9.8	— 9.9	
9	$[\text{Co}(\text{NH}_3)_4\text{CO}_3]\text{NO}_3$	— 5.7 — 8.7	— 7.7 — 12.0	— 10.0 — 15.3	— 9.9 — 15.4	Strong signal Weak signal

1) Chemical shifts measured using our spectrometer.

2) Chemical shifts measured, using Varian's V-4200 B wide line spectrometer by DHARMATI *et al.* (7). $\sigma = (H_i - H_r)/H_r$; where H_i is the resonance field for a given compound and H_r is the resonance field for the reference sample.6. - Chemical shifts of ^{17}O resonance.

The chemical shifts of ^{17}O resonance were first reported by WEAVER *et al.* (10) who investigated several compounds in a field of 10 kG using the Varian 4200B wide line spectrometer. They found the values of chemical shifts ranging from 200 mG to 6.9 G in a field of 10 kG. Barring water and concentrated nitric acid, the signals of ^{17}O in all the other compounds are very weak and the detection of such weak signals offered us an excellent opportunity for testing the sensitivity and performance of the spectrometer. We extended the study of ^{17}O resonance shifts to many organic compounds in order to interpret some of the shifts in terms of the bonding of oxygen in these compounds. The

(10) H. E. WEAVER, B. M. TOLBERT and R. LAFORCE: *Journ. Chem. Phys.*, **23**, 1956 (1955).

shifts in the compounds studied thus far in a field of 9 kG, are given in terms of σ in Table III. The organic compounds used in the experiment are the products of either B.D.H. (AnalR) or Merck (extra pure).

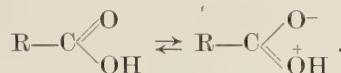
TABLE III. — *Chemical shifts of ^{17}O resonance.*

No.	Compound	$\sigma \times 10^4$	Remarks
1	Methyl alcohol	+ 0.400	—
2	Ethyl alcohol	0	98% pure
3	Water	0	Reference sample
4	Ether	0	—
5	Propionic acid	— 2.1	—
6	Acetic acid	— 2.2	—
7	Formic acid	— 2.7	—
8	Propionic anhydride	— 2.4 — 4.2	Stronger signal Weaker signal
9	Acetyl acetone	— 2.35	—
10	Ethyl acetate	— 3.1	—
11	Concentrated nitric acid . . .	— 4.6	Very strong signal
12	Acetone	— 6.0	—
13	Methyl ethyl ketone	— 6.0	—
14	Aceto acetic ester	— 6.2	Broad signal
15	Ethyl malonate	— 6.25	—

The resonance of ^{17}O in methyl and ethyl alcohols and diethyl ether is obtained close to that of water with the signal of ^{17}O in methyl alcohol occurring at a slightly higher field (about 350 mG at 9 kG). There is no measurable chemical shift of the ^{17}O the resonance in the other three compounds. These results indicate that, in compounds, in which oxygen forms a single bond with carbon or hydrogen, the resonance of ^{17}O occurs at very nearly the same field.

It is interesting to note that the resonance of ^{17}O in the two ketones studied (namely CH_3COCH_3 and $\text{CH}_3\text{COC}_2\text{H}_5$) occurs on the other hand at a considerably lower field than those observed for the aliphatic alcohols. In the ketones, the oxygen forms a double bond with carbon in which p -orbitals are used for the formation of both π and σ components. This would result in an increase in the effect due to the second order paramagnetism for the double bonded CO group as compared to that for the single bonded CO group in the alcohols. Hence one would expect the shielding of ^{17}O to be lower in the ketones than in the alcohols. No measurable chemical shift has been found between acetone and methyl ethyl ketone. This is probably because, the substitution of the ethyl group in place of the methyl group in the latter ketone, does not affect the electronic environment of the strong C = O bond appreciably.

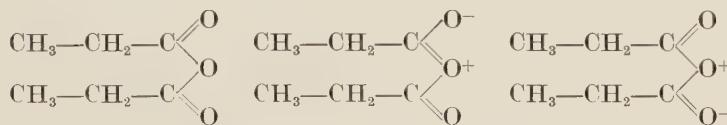
We have observed only one ^{17}O signal for the carboxylic group in formic acid, acetic acid and propionic acid. This indicates that, although this group contains two oxygens, they are structurally equivalent, owing to the possibility of a resonance between the following structures ⁽¹¹⁾ (neglecting the dimerization of acids due to hydrogen bonding),



This resonance gives the CO bond in the carboxylic group a partial double bond character. The observed bond distance for CO in the formic acid dimer ⁽¹²⁾ (1.29 Å), which lies between that of a single and double bond (1.42 Å and 1.22 Å respectively), confirms this. It may be noted here that the ^{17}O resonance for this group also occurs at fields intermediate between those for the alcohols and the ketones.

Resonance similar to the carboxylic group is also possible in esters of these acids, giving a partial double bond character to the CO group in esters. The chemical shift in ethyl acetate confirms the above reasoning.

We observed two signals of ^{17}O in propionic anhydride, with an intensity ratio of about 2:1, the stronger signal lying close to the fields of the acids while the weaker occurs at a field intermediate between those corresponding to the acids and the ketones. In the anhydride, the following type of resonance is possible ⁽¹¹⁾.



So it can be seen that there are two types of oxygens in this anhydride with one oxygen exhibiting a stronger double bond character than the other two.

The resonance of ^{17}O in acetylacetone is obtained close to the fields of resonance for the acids. It has been found by the method of bromine titration ⁽¹³⁾ that about 80% of this compound is in the enol form



⁽¹¹⁾ G. W. WHELAND: *The Theory of Resonance* (New York, 1945).

⁽¹²⁾ L. PAULING: *The Nature of Chemical Bond* (New York, 1940), p. 306.

⁽¹³⁾ E. A. BRAUDE and F. C. NACHOD: *Determination of Organic Structures by Physical Methods* (New York, 1955).

SCHOOLERY *et al.* (14) using the proton resonance estimated that about 85% is constituted in the enol form with 15% in the keto form. Our result clearly indicates that a substantial part of this compound is present in the enol form. We could not, however, detect the signal due to the other form as it would be very weak.

A preliminary investigation shows, on the other hand, that the resonance of ^{17}O in aceto-acetic ester and ethyl malonate are obtained at fields close to the fields of resonance for the ketones. This is understandable since the two compounds exist mostly in the keto form (14,15)



respectively with a slight percentage of enol form



respectively.

However, it is possible to change the relative strength of each form and then observe the signal intensities of ^{17}O due to both the forms. A check, on the amount of conversion, can then easily be made either by observing the proton resonance using a high resolution spectrometer, or by the conventional method of bromine titration. Investigation along these lines is being pursued.

It can be concluded that the study of the ^{17}O resonance yields valuable information about the bond character between carbon and oxygen in the organic compounds. In the case of single bond, it appears that the contribution to the chemical shift is predominantly due to diamagnetic effects of nearly the same magnitudes thus placing the resonance of ^{17}O at the highest field in the compounds so far studied. In the case of pure double bond, the second order paramagnetic effect is quite pronounced (due to the *p* character involved) and sometimes even overrides the overall diamagnetic effect; in the partial double bond case, the net shielding is an average of the two effects.

While we are still continuing the study of ^{17}O resonance shifts in other organic and inorganic compounds, we are also interested in their line widths which may throw more light on the structure of these compounds.

(14) J. N. SCHOOLERY, H. S. JANETT and M. S. CADLER: *Journ. Chem. Phys.*, **21**, 2092 (1953).

(15) H. B. WATSON: *Modern Theories of Organic Chemistry* (London, 1946), 2nd ed.

* * *

We wish to express our sincere thanks to Dr. C. R. KANEKAR, for his valuable suggestions regarding the study of ^{17}O resonance, to Miss S. V. NIPANAKR for preparing the cobalt complexes and to Messrs K. C. SUTARIA and V. R. MARATHE for their assistance.

RIASSUNTO (*)

Si descrive uno spettrometro ad ampia risonanza magnetica nucleare ed alta sensibilità funzionante tra 4 e 10 MHz, ed inoltre un sistema di elettromagneti di alta stabilità. Coll'aiuto dello spettrometro si sono misurati in alcuni composti gli spostamenti chimici della risonanza del ^{17}O . I risultati si interpretano in termini del legame CO in vari gruppi di composti organici.

(*) Traduzione a cura della Redazione.

K⁺-Deuterium Scattering - I (*).

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(ricevuto il 6 Dicembre 1958)

Résumé. — Nous avons étudié les diffusions élastique, inélastique et d'échange méson K⁺-deutérium à l'aide de l'approximation « impulse ». Dans ce calcul préliminaire, nous avons négligé l'interaction entre les nucléons dans l'état final. Des calculs numériques sur machine électronique 704 IBM ont été effectués dans le cas particulier où les seules amplitudes *S* méson K⁺-nucléon interviennent. Nous trouvons que la distribution angulaire de la diffusion d'échange et le rapport des sections efficaces d'échange aux sections efficaces sans échange sont les seules quantités qui peuvent permettre de déterminer facilement la longueur de diffusion dans l'état de spin isotopique *I* = 0 pour les réactions méson K⁺-neutron, quantité actuellement mal connue alors que la diffusion méson K⁺-hydrogène a permis déjà d'évaluer l'importance de l'état de spin isotopique *I* = 1.

1. — Introduction.

The K⁺-nucleon interaction appears in three basic processes: elastic scattering on protons and neutrons, exchange scattering on neutrons: strangeness conservation forbids processes involving hyperons. Since the isotopic spin of K⁺ is $\frac{1}{2}$, the three corresponding scattering amplitudes can be expressed in terms of two scattering amplitudes in the total isotopic spin states *I* = 1 and *I* = 0, and of the Coulomb scattering amplitude *f_c* (¹):

(I)	$K^+ + p \rightarrow K^+ + p$	$a_1 + f_c$
(II)	$K^+ + n \rightarrow K^+ + n$	$(a_1 + a_0)/2,$
(III)	$K^+ + n \rightarrow K^0 + p$	$(a_1 - a_0)/2.$

(*) Supported in part by the United States Air Force through the European Office, Air Research and Development Command.

(¹) It can be easily seen that coulombic phase factors must be neglected.

At the present time, our knowledge of this interaction is based on two types of experiments:

1) *Scattering of K⁺ by hydrogen* which gives information on the $I=1$ interaction. The results show a relatively constant total cross-section up to 200 MeV corresponding to a scattering length of the order of $0.24 \text{ } \hbar/\mu c$ (²). The angular distribution shows the predominance of the S state in spite of an unexplained marked minimum in the backward direction. Obviously the experimental data can be improved and make the situation clearer.

2) *Elastic and inelastic scattering of K⁺ in emulsions.* – The description of elastic scattering by an optical potential shows that the average K⁺ nucleon interaction is repulsive (³). The ratio of exchange to non exchange inelastic (or rather quasi-elastic) scattering of K⁺ should directly give the ratio of the $I=0$ and $I=1$ scattering amplitude if the nucleons, inside the nucleus, could be considered as perfectly free. This is not the case and we feel that the corrections to be done before giving the $I=0$ amplitude are rather uncertain. Nevertheless it turns out that the $I=0$ amplitude is small and seems to increase with energy.

The need for another, more direct, way of investigation of the $I=0$ interaction seems to be clear. In our opinion, the experimental data which can be handled by theoreticians are those given by scattering of K⁺ on the simplest substance containing neutrons, *i.e.* deuteron. The experiment is quite feasible, since analogous experiments have recently been done by sending K⁻ mesons in a deuterium bubble chamber (⁴).

The object of the present paper is to give a preliminary calculation of K⁺-deuteron scattering, both elastic and inelastic; this calculation uses impulse approximation; final state interactions have been provisionally neglected, and, in the numerical computation, done on IBM 704 digital computer, P -waves have been ignored. We have especially investigated the sensitivity of the results to the $I=0$ scattering length. It turns out that the angular distributions for both elastic and inelastic non exchange scattering are dominated by form factors and Coulomb scattering; therefore it would be difficult to see a modification of this angular distribution due to P waves. The most interesting quantities are the differential and total exchange cross-sections. Their ratios to the non-exchange cross-sections are very sensitive to the $I=0$

(²) J. E. LANNUTTI, S. GOLDHABER, G. GOLDHABER, W. CHUPP, S. GIAMBUZZI, C. MARCHI, G. QUARENÌ and A. WATAGHIN: *Phys. Rev.*, **109**, 2121 (1958).

(³) G. IGO, G. RAVENHALL, J. TIEMANN, W. CHUPP, G. GOLDHABER, J. LANNUTTI and R. M. THALER: *Phys. Rev.*, **109**, 2133 (1958).

(⁴) R. D. TRIPP: *Rapport Conference Genève* (1958).

amplitude and the corrections to be done to extrapolate to the case of free nucleons are probably little affected by the approximations made, especially in the high energy region.

2. – Differential cross-sections.

We bombard the deuteron in its ground state with a given distribution of momentum for the nucleons, by a K^+ meson of momentum \mathbf{k} in the laboratory system. We assume that the collisions can be treated by the impulse

approximation where the nucleons are considered as free. It is possible to characterize the scattering in terms of K^+ -nucleon amplitudes.

We neglect the binding energy of the deuteron and we have estimated the multiple scattering effects and shown that they are smaller than in the case of π meson-deuteron scattering⁽⁵⁾.

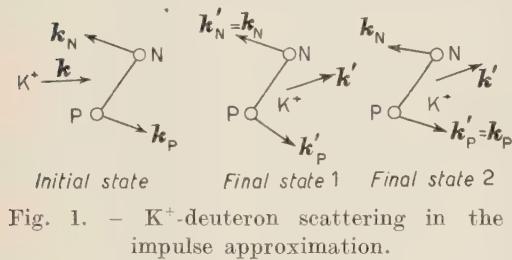


Fig. 1. – K^+ -deuteron scattering in the impulse approximation.

Let us now consider the elementary K^+ meson-nucleon processes. The amplitudes corresponding to (I), (II) and (III), can be respectively written:

$$A_p + \sigma_p \cdot \mathbf{B}_p ; \quad A_n + \sigma_n \cdot \mathbf{B}_n ; \quad A_e + \sigma_n \cdot \mathbf{B}_e ,$$

where the spin of the involved nucleon explicitly appears. We can exhibit three interesting cases:

- a) The elastic scattering where the deuteron is not dissociated by the collision.
- b) The inelastic scattering without exchange.
- c) The exchange scattering where the emerging meson is a K^0 meson. The final state is a p-p system obeying the Pauli exclusion principle.

After summation over the final spin states of the nucleon and average over the three initial spin states of the deuteron, we obtain in terms of the three K^+ -nucleon amplitudes, expressions for the cross-sections in the laboratory system corresponding to processes a), b) and c).

⁽⁵⁾ S. DRELL and L. VERLET: *Phys. Rev.*, **99**, 849 (1955).

1) *Elastic cross-section.*

$$(1) \quad \frac{d\sigma}{d \cos \theta} = 2\pi \left\{ |A_n + A_p|^2 + \frac{2}{3} |\mathbf{B}_n + \mathbf{B}_p|^2 \right\} \left| \frac{F(q)}{F(0)} \right|^2 \frac{d \cos \theta}{d \cos \theta},$$

q is the transfer moment $\mathbf{k}' - \mathbf{k}$ of the K⁺ meson; θ the angle between \mathbf{k} and \mathbf{k}' in the laboratory system and $F(q)$ the form factor defined as the Fourier transform of the square of the modulus of the deuteron radial wave function. In practical calculations we use the Hulthén wave function to describe the deuteron (see Appendix).

2) *Inelastic cross-section without exchange.* — Assuming no final state interaction we obtain in the laboratory system:

$$(2) \quad d\sigma(k, k', \cos \theta) = C \{ [|A_p|^2 + |A_n|^2 + |\mathbf{B}_p|^2 + |\mathbf{B}_n|^2] F(p, q) + \\ + [A_n A_p^* + A_p A_n^* + \frac{1}{3} (\mathbf{B}_n \mathbf{B}_p^* + \mathbf{B}_p \mathbf{B}_n^*)] G(p, q) \} \frac{p}{k} k'^2 dk' d(\cos \theta).$$

The auxiliary quantities p and q are defined by the following equations:

$$(3) \quad \begin{cases} p = k'_p - k'_n, \\ q = k' - k = -(k'_p + k'_n). \end{cases}$$

The last equality is a consequence of momentum conservation. The form factors $F(p, q)$ and $G(p, q)$ are related to the radial wave function of the deuteron; they are explicitly given in the Appendix together with the normalization constant C .

3) *Exchange cross-section.*

$$(4) \quad d\sigma(k, k', \cos \theta) = C \{ [|A_e|^2 + |\mathbf{B}_e|^2] F(p, q) - [|A_e|^2 + \frac{1}{3} |\mathbf{B}_e|^2] G(p, q) \} \cdot \\ \cdot \frac{p}{k} k'^2 dk' d(\cos \theta).$$

3. - Numerical results when K nucleon interactions occur in S state only.

In this section we restrict ourselves to spherically symmetrical energy independent elementary amplitudes a_1 and a_0 ⁽⁶⁾. With these assumptions the spin dependent scattering amplitudes B are zero and the spin independent

(6) Experimental results on K⁺-Hydrogen scattering exhibit a nearly isotropic constant cross-section up to 200 MeV⁽²⁾.

amplitudes are given by:

$$A_p = a_1 - \alpha \frac{2M}{3} \frac{1}{q^2},$$

$$A_n = \frac{a_1 + a_0}{2},$$

$$A_e = \frac{a_1 - a_0}{2},$$

where α is the fine structure constant and M the nucleon mass.

We have calculated on IBM 704 digital computer the cross-sections for $a_1 = -0.24 \text{ } \hbar/\mu c$ as given by the K^+ -hydrogen experiment analysis (2). Several values of a_0 have been tried: $a_0 = 0, 0.05, 0.06, 0.13 \text{ } \hbar/\mu c$. Various incident energies have been considered: $k = 1, 2, 2.5, 3, 3.5 \mu/\hbar c$, where μ in the π -meson mass. For the three processes involved we have computed both the differential and the total cross-sections restricted to $\cos \theta < 0.9$ because of the Coulomb effect. In the case of inelastic and exchange scattering the angular distribution of K^+ mesons is obtained by integrating over k' .

The elastic and inelastic differential cross-sections are strongly dominated by two characteristics:

- the Coulomb effects which are comparatively more important than in the π -deuteron scattering because the K -nucleon forces are about ten times smaller than the π -nucleon forces;

- the form factor $F(q)$, $F(p, q)$, $G(p, q)$; for $\theta = \pi$ the form factors considerably reduce the cross sections and the counting rate is too low.

It appears very difficult to deduce useful information on elementary amplitudes from curves such as those given in Fig. 2.

The exchange differential cross-section is more interesting because

Coulomb effects are not present. The Pauli exclusion principle reduces the forward exchange scattering because we have assumed spin independent elementary amplitudes so that the final state of the two protons system would be a

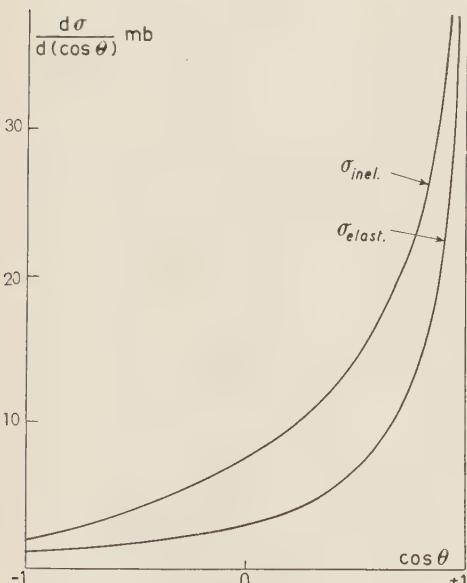


Fig. 2.

triplet state. Consequently the angular distribution exhibits a maximum from which it is, in principle, possible to deduce the value of a_0 . However, the

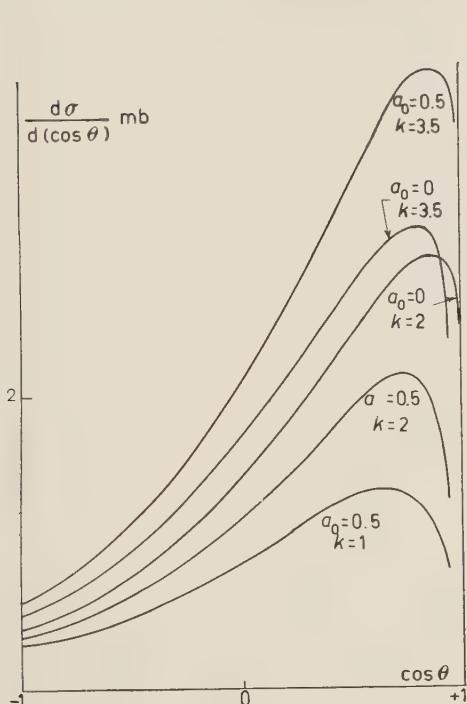


Fig. 3.

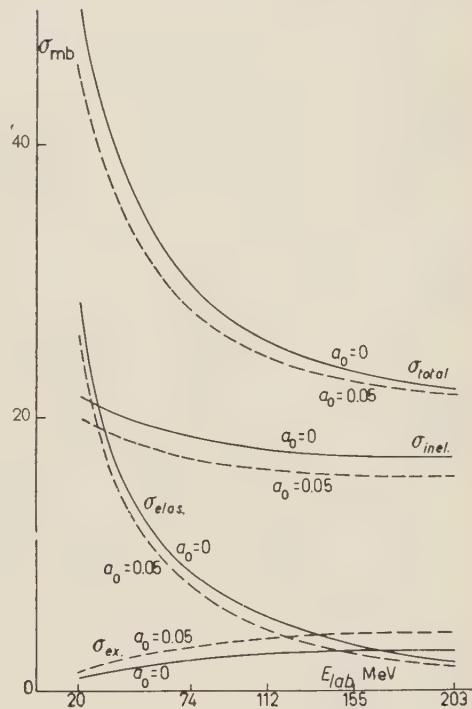


Fig. 4.

sensitivity of this maximum cross-section on the K-meson energy shows that a very good energy definition is required, which is not the case at the present time; Fig. 3 gives exchange differential cross-sections for various values of the energy and of a_0 .

In Fig. 4, we represent the variation of the total cross-sections with the energy. We note that the inelastic and exchange cross-sections are rather flat between 100 and 200 MeV; but the ratio $\sigma_{ex}/\sigma_{inel}$ increases with the energy and tends to an asymptotic value which can be computed by merely adding up the free cross-sections of K⁺ mesons on the two nucleons of the deuterium (Coulomb repulsion is negligible in

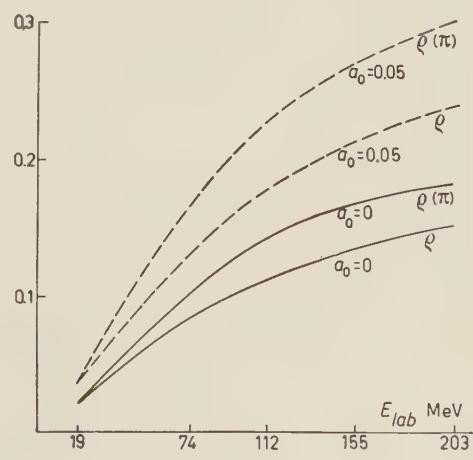


Fig. 5.

the high energy limit):

$$\frac{\sigma_{ex}}{\sigma_{n ex}} \Rightarrow \frac{(a_1 - a_0)^2}{4a_1^2 + (a_1 + a_0)^2},$$

which permits an unambiguous determination of a_0 since it is known from experiment that $|a_0| < |a_1|$.

Fig. 5 shows the energy dependence of the actual ratio; the asymptotic value is reached more rapidly by the differential ratio for $\theta = \pi$, but the experimental uncertainty would be much larger because of the very low counting rate. We feel that the deduction of the above asymptotic limit from the actual ratio is not too dependent on the approximations made. On the other hand it can be seen that this measurement does not require very good energy definition.

4. – Conclusion.

The present calculation gives encouraging results about the possibility of a theoretical interpretation of a K^+ deuteron scattering experiment. In particular we have shown that the exchange cross-section permits to draw conclusions on the $I = 0$ elementary scattering amplitude. However this preliminary work is unsatisfactory in two respects. First we should introduce phenomenologically elementary P amplitudes which are certainly present in the high energy domain. Secondly it seems necessary to take into account the interaction of the two nucleons in the final state which might enhance the cross-sections by an appreciable factor. A more careful calculation will be soon undertaken.

APPENDIX

The elastic form factor $F(q)$ is related to the deuteron radial wave function by:

$$F(q) = \frac{2}{q} \int_0^\infty |\psi_D(r)|^2 \sin \frac{qr}{2} r dr.$$

For the particular choice of the Hulthén function:

$$\psi_D(r) = \frac{1}{r} \{ \exp[-\beta r] - \exp[-\alpha r] \},$$

we obtain an analytic expression for $F(q)$:

$$\frac{F(q)}{F(0)} = \frac{1}{q} \frac{4\alpha\beta(\alpha + \beta)}{(\beta - \alpha)^2} \operatorname{arctg} \frac{16q(\beta - \alpha)^2(\beta + \alpha)}{q^4 + 4q^2[(\beta - \alpha)^2 + 2(\beta + \alpha)^2] + 64\alpha\beta(\alpha + \beta)^2}.$$

In the inelastic case, we define the Fourier transform of the wave function:

$$\varphi_D(\mathbf{k}) = \int \exp[i\mathbf{k} \cdot \mathbf{r}] \psi_D(r) d_3 r = \left[\frac{1}{k^2 + \beta^2} - \frac{1}{k^2 + \alpha^2} \right].$$

We introduce the angle between the two vectors \mathbf{p} and \mathbf{q} . By definition:

$$F(p, q) = \int_{-1}^{+1} \varphi_D(\mathbf{k}'_n) d(\cos \theta_{pq}) = \int_{-1}^{+1} \varphi_D^2(\mathbf{k}'_p) d(\cos \theta_{pq}),$$

$$G(p, q) = \int_{-1}^{+1} \varphi_D(\mathbf{k}'_n) \varphi_D(\mathbf{k}'_p) d(\cos \theta_{pq}).$$

The explicit expressions with the Hulthén function are:

$$F(p, q) = \frac{2}{(\alpha^2 + [(p+q)/2]^2)(\alpha^2 + [(p-q)/2]^2)} + \frac{2}{(\beta^2 + [(p+q)/2]^2)(\beta^2 + [(p-q)/2]^2)} -$$

$$-\frac{4}{pq(\beta^2 - \alpha^2)} \log \frac{(\alpha^2 + [(p+q)/2]^2)(\beta^2 + [(p-q)/2]^2)}{(\alpha^2 + [(p-q)/2]^2)(\beta^2 + [(p+q)/2]^2)},$$

$$G(p, q) = \frac{2(\beta^2 - \alpha^2)}{pq[\alpha^2 + \beta^2 + (p^2 + q^2)/2]} \left\{ \frac{1}{\alpha^2 + (p^2 + q^2)/4} \log \frac{\alpha^2 + [(p+q)/2]^2}{\alpha^2 + [(p-q)/2]^2} - \right.$$

$$\left. - \frac{1}{\beta^2 + (p^2 + q^2)/4} \log \frac{\beta^2 + [(p+q)/2]^2}{\beta^2 + [(p-q)/2]^2} \right\}.$$

The normalization constant C is given by:

$$C = \frac{\alpha\beta(\alpha + \beta)}{(\alpha - \beta)^2} \frac{1}{2} \frac{M}{M_r},$$

where M is the nucleon mass and M_r the reduced mass of the K^+ -meson-nucleon system.

RIASSUNTO (*)

Abbiamo studiato gli scattering elastico, anelastico e di scambio K^+ -deuterio servendoci dell'approssimazione impulsiva. In questo calcolo preliminare abbiamo trascurato l'interazione tra i nucleoni nello stato finale. Si sono effettuati calcoli numerici su macchina elettronica 704 IBM nel caso particolare che intervengano solo le ampiezze S del K^+ -nucleone. Si trova che la distribuzione angolare dello scattering di scambio e il rapporto delle sezioni di scambio efficaci alle sezioni efficaci senza scambio sono le sole grandezze che possano permettere di determinare facilmente la lunghezza di scattering nello stato di spin isotopico $I=0$ per le reazioni K^+ -neutron, grandezza attualmente mal nota, mentre lo scattering K^+ -idrogeno ha già permesso di valutare l'importanza dello spin isotopico $I=1$.

(*) Traduzione a cura della Redazione.

On the Elastic p-d Scattering at High Energies (*).

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(ricevuto l'8 Dicembre 1958)

Summary. — We have investigated the validity of the impulse approximation in the elastic p-d scattering for protons of 310 MeV. The numerical results for the differential cross-sections and polarizations as functions of the angle of scattering show clearly that this approximation is inadequate for the prediction of the additivity of these observables as it has been proved experimentally. The reason of the failure is due to the large nuclear two-body interaction in the triplet states.

Introduction.

The scattering of high energy nucleons by complex nuclei raises the question whether the experimental values of cross-sections and polarizations can be explained by means of the data corresponding to the scattering of two free nucleons.

The particular case of nucleon-deuteron scattering serves for deriving information about the continuum states of the neutron-proton system which are at present time not well known, because of the difficulty of producing a beam of monochromatic neutrons. This information is necessary to the scope mentioned before.

There are measurements ⁽¹⁾ for some types of quasi-elastic scattering which

(*) This work is a continuation of a thesis in Theoretical Physics of M. OLIVETTI (Turin, Dec. 1957).

(¹) O. CHAMBERLAIN, E. SEGRÈ, R. D. TRIPP, C. WIEGAND and T. J. YPSILANTIS: *Phys. Rev.*, **105**, 288 (1957).

clearly show additivity of the differential cross-sections and polarizations of nucleons with energies greater than 300 MeV.

At such energies one usually employs the so called impulse approximation, whose validity can be checked by investigating whether it is able to predict the experimentally observed additivity. For this reason we have calculated the values of the p-d cross-sections and polarization using the impulse approximation in the scattering of 315 MeV protons, which is known experimentally.

For the effective calculation one needs of course to know the scattering matrix of two free nucleons. Since none of the various types of phenomenological or meson theoretical interactions is able to predict, with a good accuracy, the continuum states for the two-body problem, we have used for the proton-proton scattering a set of phase shifts calculated by STAPP⁽²⁾ directly from experimental data. Since the data corresponding to neutron-proton scattering are lacking, we cannot evaluate what is left by using the impulse approximation besides the additive terms, so that we have evaluated only those terms corresponding to p-p scattering.

However, as it will appear from a more detailed examination of the remainders, the corrections due to p-n scattering do not improve the situation. These remainders are functions of the scattering angle and are important already at 15°. The situation is less favourable for the differential cross-section for which at 15° the additivity fails to be predicted with an error of nearly 14% and this error increases to nearly 22% at 90°, while for the polarization it is less than 10% between 15° and 45° and increases rapidly for larger angles. Their values are shown in Fig. 2 and 3.

For what concerns the total cross-section, it is easy to show (see Appendix) that there is a rigorous additivity if use is made of the well known theorem which gives the total cross-section as a function of the immaginary part of the scattering amplitude in the forward direction.

In Sect. 1 we express the scattering amplitude operator of the three-body problem in the total spin space as a sum of the corresponding operators of the two-body problem. This decomposition, supplied by the impulse approximation is used in Sect. 2 for evaluating the differential cross-section and the polarization in the p-d collision in terms of the two-body cross-sections and polarizations.

The scattering amplitude operator for the incoming proton $F(123)$ in the three body center of mass system is given by^(3,4)

$$(1) \quad F(123) = F'(12, 3) + F''(13, 2).$$

(2) H. P. STAPP, T. J. YPSILANTIS and N. METROPOLIS: *Phys. Rev.*, **105**, 302 (1957).

(3) G. F. CHEW: *Phys. Rev.*, **80**, 196 (1950).

(4) M. VERDE: *Handb. d. Phys.*, **39**, 143 (1957).

F' describes the scattering of particle 1 by 2 considering particle 3 as free; F'' the scattering of particle 1 by 3 while nucleon 2 is free. F' and F'' contain an average over the relative momenta of particles 2 and 3. As we shall see later, the matrix elements of the operators F' and F'' are linear combinations of those for the two-body scattering amplitude respectively. We have calculated the differential cross-section σ_{pd} and the polarization P_{pd} , following the well known formalism of the density matrix

$$(2) \quad \sigma_{pd} = \frac{\text{Tr}(F' F'^+)}{6} = \frac{1}{6} \text{Tr}(F' F'^+ + F'' F''^+ + F' F''^+ + F'' F'^+),$$

$$(3) \quad \sigma_{pd} \mathbf{P}_{pd} = \frac{1}{6} \text{Tr}[\boldsymbol{\sigma}(1) F F'^+] = \frac{1}{6} \text{Tr}[\boldsymbol{\sigma}(1)(F' F'^+ + F'' F''^+ + F' F''^+ + F'' F'^+)].$$

The $\boldsymbol{\sigma}(1) = (\sigma_x(1), \sigma_y(1), \sigma_z(1))$ in (3) are the usual Pauli spin matrices of the incoming proton. The factor $\frac{1}{6}$ is the statistical weight of the six possible spin states (3 deuteron spin states \times 2 proton spin states) which are all equally probable, since we assume no polarization in the initial states.

Neglecting the mixed products of F' and F'' , as it is requested by our approximation, (2) and (3) become

$$(2') \quad \sigma_{pd} = \frac{1}{6} \text{Tr}(F' F'^+ + F'' F''^+),$$

$$(3') \quad \mathbf{P}_{pd} = \frac{\text{Tr}\{\boldsymbol{\sigma}(1)[F' F'^+ + F'' F''^+]\}}{\text{Tr}[F' F'^+ + F'' F''^+]},$$

whence we shall deduce later

$$(2'') \quad \sigma_{pd} = \frac{16}{9} S(k, \Theta) \{ \sigma_{pp} + \sigma_{pn} + \varepsilon_{pp} + \varepsilon_{pn} \},$$

$$(3'') \quad \sigma_{pd} \mathbf{P}_{pd} = \sigma_{pp} \mathbf{P}_{pp} + \sigma_{pn} \mathbf{P}_{pn} + \boldsymbol{\eta}_{pp} + \boldsymbol{\eta}_{pn},$$

$S(k, \Theta)$ being the deuteron's form factor which is a function of the scattering angle and energy in the center of mass system of the three bodies.

The remainders are all contained in the terms $\varepsilon_{pp} + \varepsilon_{pn}$ for the scattering cross-section and in $\boldsymbol{\eta}_{pp} + \boldsymbol{\eta}_{pn}$ for the polarization.

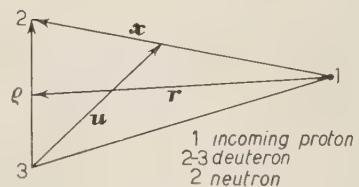


Fig. 1.

1. Reduction of the three-body scattering amplitude to the two-body one.

Let χ_J^M ($J = \frac{3}{2}, \frac{1}{2}$) be a complete orthonormal set of spin eigenfunctions in the special case when particles 2 and 3 form a deuteron. They are quite determined by the condition $\chi_J^{M*} \chi_{J'}^{M'} = \delta_{JJ'} \delta_{MM'}$ and besides by imposing them to be simultaneous eigenfunctions of the operator $\frac{1}{2}|\sigma(1) + \sigma(2) + \sigma(3)|$ with eigenvalues $\frac{3}{2}$ or $\frac{1}{2}$ and of $\frac{1}{2}|\sigma(2) + \sigma(3)|$ with eigenvalue 1. Such eigenfunctions are obtained by composition of the three triplet states of deuteron with those of particle 1.

$$\chi_J^M = \sum_{\mu=-1}^1 \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} C(1 \frac{1}{2} J; \mu m M) z_1^\mu(23) \alpha_{\frac{1}{2}}^m(1).$$

In order to reduce the three-body scattering problem to p-p and p-n scattering, it will be useful to write our eigenfunctions by means of those of the two bodies whose scattering is considered, for instance the bodies 1 and 2. The χ_J^M are so given by

$$(4) \quad \chi_J^M = \sum_{j=0}^1 \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} A_j^J C(j \frac{1}{2} J; M-m m M) z_j^{M-m}(12) \alpha_{\frac{1}{2}}^m(3) = \sum_{\substack{r=1,2 \\ s=1\dots 4}} b_{rs}^j z^s(12) \alpha^r(3) = \chi^j,$$

$$A_1^3 = 1; \quad A_0^3 = 0; \quad A_1^1 = \frac{1}{2}; \quad A_0^1 = \frac{\sqrt{3}}{2}.$$

Explicitly they are

$$(4') \quad \left\{ \begin{array}{l} J = \frac{3}{2} \\ \chi^1 = \chi_{\frac{3}{2}}^{\frac{3}{2}} = \alpha(3) T^1(12), \\ \chi^2 = \chi_{\frac{3}{2}}^{\frac{1}{2}} = \sqrt{\frac{2}{3}} \alpha(3) T^0(12) + \frac{\beta(3)}{\sqrt{3}} T^1(12), \\ \chi^3 = \chi_{\frac{3}{2}}^{-\frac{1}{2}} = \sqrt{\frac{2}{3}} \beta(3) T^0(12) + \frac{\alpha(3)}{\sqrt{3}} T^{-1}(12), \\ \chi^4 = \chi_{\frac{3}{2}}^{-\frac{3}{2}} = \beta(3) T^{-1}(12), \\ \\ J = \frac{1}{2} \\ \chi^5 = \chi_{\frac{1}{2}}^{\frac{1}{2}} = \frac{\beta(3)}{\sqrt{6}} T^1(12) - \frac{\alpha(3)}{2\sqrt{3}} T^0(12) + \frac{\sqrt{3}}{2} \alpha(3) S(12), \\ \chi^6 = \chi_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{\alpha(3)}{\sqrt{6}} T^{-1}(12) + \frac{\beta(3)}{2\sqrt{3}} T^0(12) + \frac{\sqrt{3}}{2} \beta(3) S(12), \end{array} \right.$$

$T^{\mu}(12)$ and $S(12)$ are the triplet and the singlet states of particles 1 and 2. We call:

$$\rho = \mathbf{r}_2 - \mathbf{r}_3, \quad \mathbf{r} = -\mathbf{r}_1 + \frac{\mathbf{r}_2 + \mathbf{r}_3}{2},$$

\mathbf{k}_f and \mathbf{k}_i the momenta of the incoming and outgoing proton respectively; $\varphi_d(\varrho)$ the deuteron's wave function normalized so that

$$\int |\varphi_d(\varrho)|^2 d^3\varrho = 1;$$

$\psi'(123)$ the rigorous solution of the wave equation for the three-body problem which evolves from the initial state

$$(5) \quad \exp[i\mathbf{k}_i \cdot \mathbf{r}] \varphi_d(\varrho) \chi^j(123).$$

Let $V(12)$ and $V(13)$ be the nuclear potentials which operate on space and spin co-ordinates, M the mass of the nucleon, $P(13)$ the operator which exchanges the space and the spin co-ordinates of particle 1 with those of 3.

The rigorous expression of the scattering amplitude matrix of the incoming proton, in the three-body center of mass system is

$$(6) \quad F_{ij}(123) = -\frac{1}{4\pi} \left(\frac{4}{3} \frac{M}{\hbar^2} \right) \cdot \int \chi^{*i}(123) \exp[-i\mathbf{k}_f \cdot \mathbf{r}] \varphi_d^*(\varrho) [V(12) + V(13)] (1 - P(13)) \psi^j(123) d^3\varrho d^3r.$$

We shall now derive the expression of the operator F' in (1) as a function of the p-n scattering amplitude $f(12)$.

Let $z^m(12)$ ($m=1 \dots 4$) be the spin eigenfunctions of bodies 1 and 2

$$(7) \quad z^1 = T^1(12), \quad z^2 = T^0(12), \quad z^3 = T^{-1}(12), \quad z^4 = T^0(12).$$

Let $\psi_{p_i}^m(12)$ be a complete set of rigorous solutions of the two-body wave equation, corresponding to initial states labelled by the momentum \mathbf{p}_i and the spin state m . The rigorous expression of $f_{lm}(12)$ is thus

$$(8) \quad f_{lm}(12) = -\frac{1}{4\pi} \frac{M}{\hbar^2} \int z^{l*}(12) \exp[-i\mathbf{p}_f \cdot \mathbf{x}] V(12) \psi_{p_i}^m(12) d^3x,$$

where $\mathbf{x} = \mathbf{r}_2 - \mathbf{r}_1$ and $\psi_{\rho_i}^m(12)$ evolves from the initial state

$$\exp [i\mathbf{p}_i \cdot \mathbf{x}] z^m(12) .$$

Let us define

$$g(K) = \int \exp [i\mathbf{K} \cdot \boldsymbol{\rho}] \varphi(\varrho) d^3\varrho ,$$

where \mathbf{K} is the momentum of the third particle relative to the second. With the impulse approximation we expand $\psi^i(123)$ in terms of eigenfunctions $\psi_{k_i, K}^m(x)$ of bodies 1 and 2. We neglect during the scattering process the interaction with particle 3, multiplying by a plane wave travelling in direction $\mathbf{u} = ((\mathbf{r}_2 + \mathbf{r}_1)/2) - \mathbf{r}_3$ and whose momentum is $\mathbf{k}_u = \mathbf{K} - (\mathbf{k}_i/2)$, so that the scattering amplitude operator depends upon the initial state, with momentum \mathbf{k}_i .

$\psi_{k_i, K}^m(x)$ is a function of the momentum \mathbf{k}_i of the incident nucleon and of \mathbf{K} . The last one may assume all possible values according to the momentum distribution in the deuteron. Therefore

$$(9) \quad \psi'^j = \sum_{\substack{m=1,2 \\ l=1\dots 4}} \int g(K) \psi_{k_i, K}^l(x) a_{ml}^j \alpha^m(3) \exp [i\mathbf{k}_u \cdot \mathbf{u}] d^3 K ,$$

where $\alpha^m(3)$ are the spin eigenfunctions of the third particle. The coefficients a_{ml}^j are defined by the condition that $\psi^i(123)$ approaches the initial state for $r \rightarrow \infty$, i.e.

$$(9') \quad \psi'^j(123) \underset{r \rightarrow \infty}{\sim} \int g(K) \exp \left[i(\lambda \mathbf{K} + \mu \mathbf{k}_i)(\mathbf{r}_2 - \mathbf{r}_1) + i \left(\mathbf{K} - \frac{\mathbf{k}_i}{2} \right) \left(-\mathbf{r}_3 + \frac{\mathbf{r}_2 + \mathbf{r}_1}{2} \right) \right] d^3 \varrho d^3 r d^3 K \cdot \left[\sum_{\substack{m=1,2 \\ l=1\dots 4}} a_{ml}^j \alpha^m(3) \alpha^l(12) \right] .$$

One finds then

$$\lambda = \frac{1}{2}, \quad \mu = \frac{3}{4},$$

therefore

$$(10) \quad \psi_{k_i, K}^m(x) \underset{r \rightarrow \infty}{\sim} \exp [i\frac{1}{2}\mathbf{K} \cdot \mathbf{x} + i\frac{3}{4}\mathbf{k}_i \cdot \mathbf{x}] z^m(12) .$$

Assuming as new variables \mathbf{x} and $\boldsymbol{\rho}$ we have $\mathbf{u} = -(\mathbf{x}/2) + \boldsymbol{\rho}$; $\mathbf{r} = \mathbf{x} - (\boldsymbol{\rho}/2)$

and in account of (4)

$$(11) \quad F'_{ij} = -\frac{1}{4\pi} \left(\frac{4}{3} \frac{M}{\hbar^2} \right) S^{\frac{1}{2}}(k, \Theta) \sum_{\substack{m,r=1,2 \\ l,s=1\dots 4}} b_{rs}^i a_{ml}^j \delta_{rm} \cdot \\ \cdot \int \exp[-ik_f \cdot \mathbf{x}] z^s(12) V(x) \psi_{k_i,0}^l(x) \exp[i\mathbf{k}_i \cdot \frac{\mathbf{x}}{4}] d^3x,$$

where

$$S^{\frac{1}{2}}(k, \Theta) = \int g^* \left(\left| \frac{\mathbf{k}_f - \mathbf{k}_i}{2} + \mathbf{K} \right| \right) g(K) d^3K,$$

is the deuteron's form factor and Θ the scattering angle in the three-body center of mass system.

In eq. (11) we have taken into account the fact that $\psi_{k_i,K}^l$ gives a contribution in the integral over K only in the neighbourhood of $K = 0$, and we have neglected the integral containing the exchange of coordinates of nucleon 1 with those of 3, which does not contribute in the forward direction and is unimportant at large energies and at small angles.

From

$$\psi_{k_i,0}^l \underset{x \rightarrow \infty}{\sim} \exp[i\frac{3}{4}\mathbf{k}_i \cdot \mathbf{x}] z^l(12)$$

it follows

$$\psi_{k_i,0}^l \exp[i\frac{1}{4}\mathbf{k}_i \cdot \mathbf{x}] \underset{x \rightarrow \infty}{\sim} \exp[i\mathbf{k}_i \cdot \mathbf{x}] z^l(12),$$

and in account of (7)

$$(12) \quad F'_{ij} = \frac{4}{3} S^{\frac{1}{2}} \sum_{\substack{r=1,2 \\ s,l=1\dots 4}} b_{rs}^i a_{rl}^j f_{sl}^{\text{np}}(12).$$

We have now to evaluate the coefficients $\sum_r b_{rs}^i a_{rl}^j$.

From (9') it is clear that $a_{rl}^j = b_{rl}^j$, so that

$$\sum_r b_{rs}^i a_{rl}^j = \sum_r b_{rs}^i b_{rl}^j.$$

Since

$$b_{rs}^i = |\alpha^r(3)| |z^s(12)| \chi^i(123) = |\chi^i(123)| |\alpha^r(3)| |z^s(12)|$$

and

$$\sum_r |\alpha^r(3)| |\alpha^r(3)| = 1$$

we obtain

$$(13) \quad \sum_r b_{rs}^i b_{rl}^j = \langle \langle \chi^i(123) | z^s(12) \rangle | \langle z^l(12) | \chi^j(123) \rangle \rangle.$$

The corresponding numerical values are thus derived from Eqs. (4') and (6'). The scattering process of 1 by 3, is evaluated in an analogous manner. The wave function $\psi(123)$ becomes

$$(14) \quad \psi''^j(123) = \sum_{\substack{m=1,2 \\ l=1\dots 4}} \int g(K) \psi_{k_z K}^l(\mathbf{r}_3 - \mathbf{r}_1) b_{ml}^j \alpha^m(3) \exp[i\mathbf{k}_w \cdot \mathbf{u}'] ,$$

where

$$\mathbf{k}_w = -\mathbf{K} - \frac{\mathbf{k}_i}{2} .$$

The term of the scattering amplitude which does not contain the permutation of 1 and 3 is reducible, as in the preceding case, to an integral of type (12), while the term containing $P(13)$ gives the following integral

$$(I)_{ij} = -\frac{4}{3} S^{\frac{1}{3}} \sum_{\substack{m,r=1,2 \\ s,l=1\dots 4}} b_{rs}^i b_{ml}^j \delta_{rm} \left(-\frac{1}{4\pi} \frac{M}{\hbar^2} \right) \int \exp \left[-i \frac{3}{4} \mathbf{k}_f \cdot \mathbf{x} \right] V(x) \psi_{k_i 0}^l(-x) d^3x ,$$

using the new variables

$$\mathbf{y} = \mathbf{r}_2 - \frac{\mathbf{r}_1 + \mathbf{r}_3}{2} , \quad \mathbf{x} = \mathbf{r}_3 - \mathbf{r}_1 ,$$

$\psi_{k_i 0}^l(13)$ is the function which evolves from $\exp[i\frac{3}{4}\mathbf{k}_i \cdot \mathbf{x}]z'(13)$, as follows from the condition at infinity for ψ''^j .

The sum of the two integrals gives

$$(15) \quad F''_{ij}(123) = \frac{4}{3} S^{\frac{1}{3}}(k, \Theta) \sum_{\substack{r=1,2 \\ s,l=1\dots 4}} b_{rs}^i b_{rl}^j f_{sl}^{(pp)}(13) ,$$

$f_{sl}(13)$ are the matrix elements for the scattering amplitude of two nucleons, no matter if they are identical or not, whose initial and final momenta \mathbf{p}_i and \mathbf{p}_f are related to \mathbf{k}_i and \mathbf{k}_f by (4)

$$(16) \quad \begin{cases} \mathbf{k}_i - \mathbf{k}_f = \mathbf{p}_i - \mathbf{p}_f , \\ \frac{3}{4} \cdot (\mathbf{k}_i + \mathbf{k}_f) = \mathbf{p}_i + \mathbf{p}_f . \end{cases}$$

2. – Evaluation of the differential cross section and polarization.

We can now proceed to calculate the proton-deuteron cross-section and polarization as a function of those for p-p and p-n processes.

The two-body scattering amplitude in order to satisfy the usual condition

of invariance by transformations of Galilei and space an time reflections, must have in the spin space the following expression ⁽¹⁾

$$(17) \quad f = BS + C(\sigma_{(1)} + \sigma_{(2)}) \cdot \mathbf{n} + \frac{1}{2}G[(\sigma_{(1)} \cdot \mathbf{q})(\sigma_{(2)} \cdot \mathbf{q}) + (\sigma_{(1)} \cdot \mathbf{P})(\sigma_{(2)} \cdot \mathbf{P})]T + \\ + \frac{1}{2}H[(\sigma_{(1)} \cdot \mathbf{q})(\sigma_{(2)} \cdot \mathbf{q}) - (\sigma_{(1)} \cdot \mathbf{P})(\sigma_{(2)} \cdot \mathbf{P})]T + N(\sigma_{(1)} \cdot \mathbf{n})(\sigma_{(2)} \cdot \mathbf{n})T,$$

where

$$\mathbf{n} = \frac{\mathbf{p}_f \wedge \mathbf{p}_i}{|\mathbf{p}_f \wedge \mathbf{p}_i|}; \quad \mathbf{q} = \frac{\mathbf{p}_f - \mathbf{p}_i}{|\mathbf{p}_f - \mathbf{p}_i|}; \quad \mathbf{P} = \frac{\mathbf{p}_f + \mathbf{p}_i}{|\mathbf{p}_f + \mathbf{p}_i|},$$

T and S are the triplet and singlet projection operators

$$T = \frac{1}{4}(\sigma_{(1)} \cdot \sigma_{(2)} + 3), \quad S = \frac{1}{4}(1 - \sigma_{(1)} \cdot \sigma_{(2)}).$$

The complex coefficients B , C , G , H , N , are scalar functions of the scattering angle $(\mathbf{p}_i, \mathbf{p}_f) = \theta$ and of energy.

Ordering the matrix f according to the spin eigenfunctions z^m (12) and choosing and z as y axes the directions of \mathbf{P} and \mathbf{n} respectively ⁽⁵⁾ we have

$$(18) \quad f = \begin{pmatrix} \frac{1}{2}(G - H \cos \theta) & -\sqrt{2}(iC + \frac{1}{2}H \sin \theta) & \frac{1}{2}(G + H \cos \theta) - N & 0 \\ \sqrt{2}(iC - \frac{1}{2}H \sin \theta) & N + H \cos \theta & -\sqrt{2}(iC - \frac{1}{2}H \sin \theta) & 0 \\ \frac{1}{2}(G + H \cos \theta) - N & \sqrt{2}(iC + \frac{1}{2}H \sin \theta) & \frac{1}{2}(G - H \cos \theta) & 0 \\ 0 & 0 & 0 & B \end{pmatrix}.$$

The operator f preserves the total spin, so that the elements $f_{\alpha 4}$ and $f_{4\alpha}$ with $\alpha \neq 4$ vanish identically in both p-p and p-n cases. In the p-n case that is a consequence of the fact that the term $(\sigma_{(1)} - \sigma_{(2)}) \cdot \mathbf{n}$ is absent; its presence would imply a charge-dependence of nuclear forces. The ceofficients B , C , G , H , N are different in the two cases, for the necessary antisymmetrization required bu Pauli's principle. The matrix F' of (1) and (11), and so F'' , is connected to $f(12)$ by the following linear relations

$$(19) \quad \langle \chi_{J'}^{M'} | F' | \chi_J^M \rangle = \sum_{mj} A_j^{J'} A_j^J C(j \frac{1}{2} J'; M' - m \ m \ M') C(j \frac{1}{2} J; M - m \ m \ M) \cdot \\ \cdot \langle j \ M' - m | f(12) | j \ M - m \rangle$$

as one can easily deduce from (4), (12) and (13).

(5) With this choice we have $|\text{Tr } (\sigma_{(1)}FF^+)| = |\text{Tr } (\sigma_{(2)}FF^+)|$ since the polarization must be normal to the scattering plane; see L. WOLFENSTEIN and J. ASHKIN: *Phys. Rev.*, **85**, 947 (1952).

It is clear from (17) that f is a sum of terms each one expressible as a scalar product of a spin tensor of well defined rank with a space tensor of the same rank. Introducing these tensor components Θ_L^M ($L = 0, 1, 2$), we have

$$(20) \quad \langle jm' | f | jm \rangle = \sum_{L=0}^2 \sum_{\mu=-L}^L B_{Lj}^\mu \langle jm' | \Theta_L^\mu | jm \rangle .$$

From (17) one derives

$$(20') \quad \begin{cases} B_0^0 = BY_0^0(\theta\varphi), & B_{01}^0 = \frac{1}{3}(G+N)Y_0^0(\theta\varphi), \\ B_{11}^\mu = CY_1^{-\mu}(\mathbf{n})(-1)^\mu, \\ B_{21}^\mu = \left\{ \frac{1}{6}(G+N)Y_2^{-\mu}(\mathbf{q}) + \frac{1}{6}(G-H)Y_2^{-\mu}(\mathbf{P}) + \frac{1}{3}N Y_2^{-\mu}(\mathbf{n}) \right\}(-1)^\mu, \end{cases}$$

$$(20'') \quad \begin{cases} \langle 00 | \Theta_0^0 | 00 \rangle = \langle 10 | \Theta_0^0 | 10 \rangle = \sqrt{4\pi}, \\ \langle 10 | \Theta_1^0 | 10 \rangle = 2 \frac{\sqrt{4\pi}}{\sqrt{3}}, \\ \langle 10 | \Theta_2^0 | 10 \rangle = 2 \frac{\sqrt{4\pi}}{\sqrt{5}}, \end{cases}$$

Y_L^μ are the spherical harmonics normalized to one on the sphere of unity radius

$$Y_L^\mu = \frac{(-1)^{L+\mu}}{2^L L!} \left[\frac{(2L+1)(L-\mu)!}{4\pi(L+\mu)!} \right]^{\frac{1}{2}} (\sin \theta)^\mu \left(\frac{\partial}{\partial \cos \theta} \right)^{L+\mu} (\sin \theta)^{2L} \exp[i\mu\varphi].$$

Making use of Wigner-Eckart's theorem (6) for every matrix element one obtains

$$(21) \quad \langle jm' | f | jm \rangle = \sum_{L\mu} B_{Lj}^\mu C(jLj; m\mu m') \langle j | \Theta_L | j \rangle ,$$

and hence one derives the well known expression for the differential cross-section and for the polarization in the two-body scattering (see (2') and (3'))

$$(22) \quad \sigma(12) = \frac{1}{4} T(f\bar{f}^+) = \frac{1}{4} \sum_{L\mu} \frac{2j+1}{2L+1} (-1)^\mu B_{Lj}^\mu B_{Lj}^{-\mu} |\langle j | \Theta_L | j \rangle|^2 ,$$

(6) See for instance, ROSE: *Elementary Theory of Angular Momentum*, p. 85.

$$(22') \quad \sigma(12) P(12) = \frac{1}{4} \text{Tr} (\sigma_y(1) f f^+) = \\ = \frac{i}{\sqrt{2}} \frac{1}{4} \sum_{L\mu} \sum_{\mu'} \frac{(2j+1)^{\frac{1}{2}}}{(2L+1)^{\frac{1}{2}}} (-1)^{\mu} B_{Lj}^{-\mu} W(1jLj; jL') \cdot \\ \cdot \{ B_{Lj}^{\mu} C(1L'L; -1\mu'\mu' -1) + B_{Lj}^{\mu'} C(1L'L; 1\mu'\mu' -1) \} j_1 \sigma_1 j_2 \langle j | \Theta_L | j \rangle j' \Theta_{L'} | j' \rangle^*,$$

where the $W(j_1 j_2 j j_3; j' j'')$ are the Racah coefficients ⁽⁷⁾.

Similarly using Wigner-Eckart's theorem for the matrix elements of F' one obtains

$$(23) \quad \langle J' M' | F' | JM \rangle = \sum_{L\mu} D_{LJJ'}^\mu C(JLJ'; M\mu M') \langle J' | \Xi_L | J \rangle,$$

where $D_{LJJ'}^\mu$ and Ξ_L are quantities corresponding to B_{Lj}^μ and Θ_L .

The differential cross-section $\sigma'(123)$ which is referred to the scattering process by one of the nucleons in the deuteron is thus

$$(24) \quad \sigma'(123) = \frac{1}{6} \text{Tr} (F' F'^+) = \frac{1}{6} \sum_{J'J} \sum_{L\mu} (-1)^\mu D_{LJJ'}^\mu D_{LJJ'}^{-\mu} \frac{2J'+1}{2L+1} |\langle J' | \Xi_L | J \rangle|^2.$$

By comparison of (23) with (19) one obtains

$$(25) \quad D_{LJJ'}^\mu \langle J' | \Xi_L | J \rangle = \\ = \sum_j A_j^J A_j^{J'} B_{Lj}^\mu (-1)^{J-J'} (2J+1)^{\frac{1}{2}} (2j+1)^{\frac{1}{2}} W(LjJ'\frac{1}{2}; jJ) \langle j | \Theta_L | j \rangle.$$

Consequently $\sigma'(123)$ becomes

$$(26) \quad \sigma'(123) = \frac{1}{6} \sum_{J'J} \sum_{L\mu} A_j^J A_j^{J'} A_{j'}^{J'} A_{j'}^J \frac{(2j+1)^{\frac{1}{2}} (2j'+1)^{\frac{1}{2}}}{(2L+1)^{\frac{1}{2}}} (2J'+1) (2J+1) \cdot \\ \cdot (-1)^\mu B_{Lj}^\mu B_{Lj'}^{-\mu} W(LjJ'\frac{1}{2}; jJ) W(Lj'J'\frac{1}{2}; j'J) \langle j | \Theta_L | j \rangle \langle j' | \Theta_L | j' \rangle^*,$$

and separating the contribution of the terms with $j \neq j'$ from those with $j = j'$

$$(26') \quad \sigma'(123) = \frac{2}{3} \sum_{J'J} (2J+1) (2J'+1) \sum_{L\mu} W^2(LjJ'\frac{1}{2}; jJ) \frac{1}{4} (A_j^J)^2 (A_{j'}^{J'})^2 \frac{2j+1}{2L+1} \cdot \\ \cdot (-1)^\mu B_{Lj}^{-\mu} B_{Lj'}^\mu |\langle j | \Theta_L | j \rangle|^2 + \\ + \frac{\sqrt{3}}{4} W(01\frac{1}{2}\frac{1}{2}; 1\frac{1}{2}) W(00\frac{1}{2}\frac{1}{2}; 0\frac{1}{2}) \text{Re} \{ B_{01}^0 B_{00}^{0*} \langle 1 | \Theta_0 | 1 \rangle \langle 0 | \Theta_0 | 0 \rangle^* \}.$$

⁽⁷⁾ See for instance, ROSE: *Elementary Theory of Angular Momentum*, p. 108.

(We have made use here of the triangular inequalities for the Racah coefficients and of the values of A_j^J .)

The first term at the right hand side of (26') contains, as one can easily see, just the same four terms with the same tensor ranks appearing in $\sigma(12)$, with different coefficients.

By comparison of (26') with (22) and denoting by $C_j^L(12)$ the coefficients of $|\langle j | \Theta_L | j \rangle|^2$ in $\sigma(12)$, one finds

$$(27) \quad \begin{aligned} \sigma'(123) = & \sigma(12) + \frac{1}{8} \operatorname{Re} \{ B_{01}^0, B_{00}^{0*} \langle 1 | \Theta_0 | 1 \rangle \langle 0 | \Theta_0 | 0 \rangle^* \} - \\ & - \frac{1}{4} C_0^0(12) |\langle 0 | \Theta_0 | 0 \rangle|^2 - \frac{1}{12} C_1^0(12) |\langle 1 | \Theta_0 | 1 \rangle|^2 - \\ & - \frac{1}{6} C_1^1(12) |\langle 1 | \Theta_1 | 1 \rangle|^2 - \frac{1}{3} C_1^2(12) |\langle 1 | \Theta_2 | 1 \rangle|^2. \end{aligned}$$

We can therefore assert that, apart from the first term, the others are all comprised between 9% and 33% of the corresponding ones in $\sigma(12)$.

It is noteworthy here to call attention on the preponderance of the term $-\frac{1}{6} C_1^1(12) |\langle 1 | \Theta_1 | 1 \rangle|^2$ which practically dominates the remainder. This means that the impulse approximation is inadequate as soon as strong two-body interactions are effective. The amplitude $\langle 1 | \Theta_1 | 1 \rangle$ refers indeed to the triplet substate. Proceeding in the same way, the polarization results to be

$$(28) \quad \begin{aligned} \frac{1}{6} \operatorname{Tr} (\sigma_y(1) F' F'^+) = & \frac{i}{6\sqrt{2}} \sum_{JJ''} \sum_{LL'} \sum_{j_1 j_2 j_3} A_j^J A_{j_1}^{J'} A_{j_2}^{J''} A_{j_3}^J A_{j_1}^{J''} A_{j_2}^{J'} A_{j_3}^{J''} \frac{(2j+1)^{\frac{1}{2}} (2J+1)}{(2L'+1)^{\frac{1}{2}}} \cdot \\ & \cdot (2J''+1)(2j_1+1)^{\frac{1}{2}} (2j_2+1)^{\frac{1}{2}} (-1)^{J+j_1+1} W(1jJ'\frac{1}{2}; j'J) W(1JL'J''; J'L) \cdot \\ & \cdot W(Lj_1J'\frac{1}{2}; j_1J'') W(L'j_2J'\frac{1}{2}; j_2J'') \langle j_1 | \Theta_L | j_1 \rangle^* \langle j_2 | \Theta_L | j_2 \rangle \langle j' | \sigma_1 | j \rangle \cdot \\ & \cdot [C(1LL'; -1\mu\mu-1) B_{Lj_2}^{\mu-1} + C(1LL'; 1\mu\mu+1) B_{Lj_2}^{\mu+1}] (-1)^\mu B_{Lj_1}^{-\mu}. \end{aligned}$$

With reference to (2'') and (3''), we give here the expression of the cross-section and the polarization for the two-body scattering, and the remainders

$$(29) \quad \left\{ \begin{array}{l} \sigma(12) = \frac{1}{4} |B|^2 + 2|C|^2 + \frac{1}{2} |H|^2 + \frac{1}{2} |N|^2 + \frac{1}{4} |G-N|^2, \\ \varepsilon = -\frac{1}{3} |C|^2 - \frac{1}{16} (|B|^2 + |G|^2) - \frac{1}{6} |H|^2 - \frac{11}{48} |N|^2 + \\ \qquad + \frac{5}{24} \operatorname{Re} (GN^*) + \frac{1}{24} \operatorname{Re} [(N+G)B^*], \end{array} \right.$$

$$(30) \quad \left\{ \begin{array}{l} \sigma(12) P(12) = 2 \operatorname{Re} (CN^*), \\ \eta = \frac{1}{6} \operatorname{Im} \left\{ i(G+B-3N)C^* - \frac{1}{6} HN^* \sin \theta \right\}. \end{array} \right.$$

In the numerical calculations of ε_{pd} and η_{pd} and their percentage with respect to the additive terms, we have used the set No. 4 of phase shifts in p-p scattering, as already mentioned (2). The choice of set No. 4 is due to practical reasons, since a large number of tables and curves given by Stapp, are referred to this set.

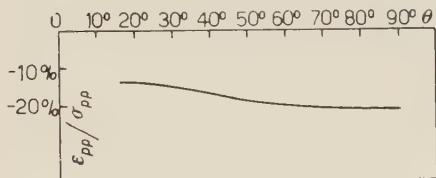


Fig. 2.

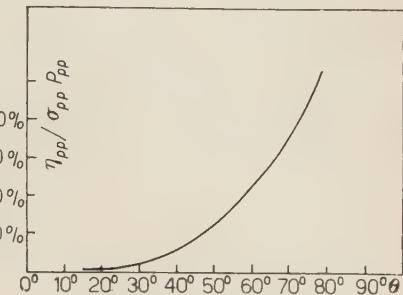


Fig. 3.

However, as it is shown in Stapp's work, the sets No. 1, No. 2, No. 3, No. 6 are also good for what concerns the values of the cross-section and polarization. The percentage value of the remainders ε_{pd} and η_{pd} are plotted as functions of the scattering angle θ in Fig. 2 and 3.

* * *

We wish to thank Prof. M. VERDE for suggesting this problem and for his kind encouragement and many useful discussions.

APPENDIX

Additivity of the total cross sections.

We shall prove the rigorous additivity of the total cross section σ^{tot} , by help of the well known theorem relating the imaginary part of the scattering amplitude in the forward direction, to the total cross section

$$(31) \quad \sigma_{pd}^{(tot)} = \frac{4\pi}{k} \left(\sum_{ij} F_{ij} \right)_{k_f = k_i}.$$

Since $\frac{d}{d\theta} \text{tg } \Theta/2 = \text{tg } \theta/2$ and, when $\theta = 0$, $p = \frac{1}{2}k$ as one can see from eq. (16), it is easy to deduce that

$$(32) \quad \sigma_{pd}^{(tot)} = \sigma_{pn}^{tot} + \sigma_{pp}^{tot}.$$

It is just sufficient to bear in mind the expansion in terms of associated Legendre polynomials ⁽²⁾ of the f_{lm} . It is then clear that, in the forward direction, f_{13} , f_{31} , f_{12} , f_{32} , f_{21} , f_{23} vanish, since

$$f_{13} = f_{31} = \sum_l a^l P_l^0(\theta); \quad f_{12} = -f_{32} = \sum_l b^l P_l^1(\theta); \quad f_{21} = -f_{23} = \sum_l c^l P_l^1(\theta),$$

where a^l , b^l , c^l depend upon the phase shifts and the energy. Moreover, it can be seen directly from the matrix (18) itself, that the sum of the last four elements vanishes. Since,

$$\sigma_{(12)}^{\text{tot}} = \frac{4\pi}{p} \operatorname{Im} \left(\sum_{ij} \frac{f_{ij}}{4} \right)_{p_f = p_i},$$

and so for $\sigma_{(13)}^{\text{tot}}$, it follows

$$\sigma_{pd}^{\text{tot}} = \frac{4\pi}{p} \sum_i \operatorname{Im} \frac{f_{ii}^{pn}(0)}{4} + \frac{4\pi}{p} \sum_i \operatorname{Im} \frac{f_{ii}^{pp}(0)}{4},$$

q.e.d.

RIASSUNTO

Si è studiata la validità dell'approssimazione impulsiva, nel caso dell'urto elastico p-d a 310 MeV. I valori numerici delle sezioni d'urto differenziali e delle polarizzazioni, come funzioni dell'angolo d'urto, mostrano chiaramente che questa approssimazione non è in grado di prevedere, per queste osservabili, l'additività verificata sperimentalmente. Tale insufficienza deve attribuirsi alla forte interazione nucleare tra due corpi negli stati di tripletto.

Direct Decay $\pi^0 \rightarrow e^+ + e^-$.

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(ricevuto il 9 Dicembre 1958)

Summary. — The direct decay $\pi^0 \rightarrow e^+ + e^-$ is discussed and analyzed as a probe of « pion size ».

The decay process $\pi^0 \rightarrow e^+ + e^-$ is of interest as a probe of pion « size » in the following sense. Think of the decay proceeding via two intermediate photons (1) according to the interaction

$$(1) \quad \mathcal{L}_{\text{int}} = \lambda \int d^3x \varphi_\pi(x) \mathbf{E}(x) \cdot \mathbf{B}(x) .$$

The photons then convert as shown in Feynman diagram 1.

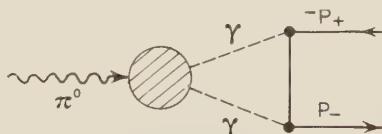


Fig. 1.

Aside from known and calculable electromagnetic factors describing the conversion of the photons to an $e^+ e^-$ pair, the ratio of the decay rates

$$(2) \quad r = \frac{\text{Rate } (\pi^0 \rightarrow e^+ + e^-)}{\text{Rate } (\pi^0 \rightarrow \gamma + \gamma)} ,$$

(*) Ford Foundation Fellow on leave from Stanford University, Stanford, California, summer, 1958.

(1) In Eq. (1) φ_π is the pion-field amplitude, and $E \cdot B$ the gauge invariant pseudo-scalar formed from the electric and magnetic field strengths E and B .

depends only upon the « structure » of the interaction as summarized, to indicate a special example, by an invariant structure factor $I[(x-y)^2]$ in

$$(1') \quad \mathcal{L}'_{\text{int}} = \lambda \int d^3x \varphi_\pi(x) \int d^4y I[(x-y)^2] \mathbf{E}(x) \cdot \mathbf{B}(y).$$

This is because the emission of virtual intermediate photons of large four momentum is inhibited by the structure factor I . On the other hand the real photons in the $\pi^0 \rightarrow \gamma + \gamma$ decay satisfy the Einstein condition $k_\mu k^\mu = 0$ and thus have no invariant « length » with which to probe the structure in (1').

Indeed, in the limit of a point interaction (1), the Feynman amplitude for diagram I diverges logarithmically ⁽²⁾. It is anticipated, however, that the « size » of this vertex will be \approx nucleon Compton wavelength $(1/M)$ since nucleon-antinucleon pair annihilation into photons is usually held to be the mechanism underlying (1). If the Feynman amplitude for $\pi^0 \rightarrow e^+e^-$ is calculated with (1), and convergence is achieved with a Bopp-Feynman-Podolsky regulator corresponding to a mass M , it is found directly that the branching ratio (2) is $r \approx 10^{-7}$.

In order to proceed further and study more generally the dependence of r on the structure of (1'), it is advantageous to turn to a spectral representation for this vertex. The use of spectral representations for the vertex operator has been discussed in a number of recent communications ⁽³⁾; we shall only outline its application here.

We are interested in the matrix element and hence form factor defined by

$$(3) \quad \langle p_+ p_- | j_\pi(0) | 0 \rangle = (4E_{p_+} E_{p_-})^{-\frac{1}{2}} (\bar{u}_{p_-} | F[(p_+ + p_-)^2] \gamma_5 | v_{p_+}).$$

F is given by a dispersion relation

$$(4) \quad F(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im } F(\sigma^2)}{\sigma^2 - Q^2 - i\epsilon} d\sigma^2; \quad Q \equiv p_+ + p_-.$$

⁽²⁾ Interaction (1) is not renormalizable because of the appearance of field strengths as required by electromagnetic gauge invariance.

⁽³⁾ G. F. CHEW, R. KARPLUS, S. GASIOROWICZ and F. ZACHARIASEN: *Phys. Rev.*, **110**, 265 (1958); Y. NAMBU: *Nuovo Cimento*, **9**, 610 (1958); R. OEHME: *Phys. Rev.*, **111**, 1430 (1958); M. GOLDBERGER and S. TREIMAN: *Phys. Rev.*, **110**, 1178 (1958); **111**, 354 (1958); P. FEDERBUSH, M. GOLDBERGER and S. TREIMAN: *Phys. Rev.*, **112**, 642 (1958); S. D. DRELL and F. ZACHARIASEN: *Phys. Rev.*, **111**, 1727 (1958). The notation of this reference is used here.

in order to evaluate $\text{Im } F(\sigma^2)$ we turn to the absorptive amplitude

$$(5) \quad A(p_+, p_-) = (\bar{u}_{p_-} | \gamma_5 \text{Im } F(Q^2) | v_{p_+}) = \\ = - (E_{p_+}/2)^{+ \frac{1}{2}} \sum_n (2\pi)^4 \delta^4(p_n - Q) \langle p_+ | \bar{u}_{p_-} j_e(0) | n^\pm \rangle \langle n^\pm | j_\pi(0) | 0 \rangle$$

and find, with neglect of all but the two-photon contribution to \sum_n and with use of Born approximation for the electron-photon Compton scattering amplitude,

$$(6) \quad \text{Im } F(Q^2) = \left(\frac{\alpha \lambda}{4(2m_\pi)^{\frac{1}{2}}} m_e \ln \frac{4Q^2}{m_e^2} \right) G(Q^2),$$

The structure factor $G(Q^2)$ is defined by the matrix element to a two-photon state (momenta q_1 and q_2 and polarization ε_1 and ε_2 ; $\varepsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor of rank four):

$$(7) \quad \langle q_1 q_2^- | j_\pi(0) | 0 \rangle = ((\lambda/2)(8m_\pi q_1 q_2)^{\frac{1}{2}}) \varepsilon_{1\mu} q_{1\nu} \varepsilon_{2\sigma} q_{2\mu} \varepsilon^{\mu\nu\rho\sigma} G((q_1 + q_2)^2).$$

Equation (6) in (3) and (4) leads to the branching ratio (2)

$$(8) \quad r = \frac{1}{2} \left(\frac{\alpha}{\pi} \frac{m_e}{m_\pi} \right)^2 \left| \int_0^\infty \frac{dx}{x-1-i\varepsilon} G(xm_\pi^2) \ln \frac{m_\pi^2}{m_e^2} x \right|^2,$$

which depends only on the space-time structure of the vertex (1'). The factor σ^2 in this ratio is due to the additional electromagnetic vertices, and a factor of the form $(m_e/E) \ln E/m_e$ is typical for high-energy electrodynamic amplitudes; in this case E corresponds to the pion rest energy.

We give a few numerical results in order to show the sensitivity of ratio r to pion size (4). Choosing for simplicity

$$G(xm_\pi^2) = \begin{cases} 1, & x < p^2 \\ 0, & x > p^2 \end{cases}$$

we integrate (8) to (5)

$$r = \frac{1}{2} \left(\frac{\alpha}{\pi} \frac{m_e}{m_\pi} \right)^2 \left\{ \left[\ln p^2 \ln \frac{m_\pi^2 p}{m_e^2} \right]^2 + \left[\pi \ln \frac{m_\pi^2}{m_e^2} \right]^2 \right\},$$

(4) Evaluation of G in a dispersion theoretic study of $\pi^0 \rightarrow \gamma + \gamma$ decay is discussed in M. L. GOLDBERGER and S. B. TREIMAN: *Nuovo Cimento*, **9**, 451 (1958).

(5) In a perturbation calculation which assumes that the π^0 decays via a closed nucleon loop, the ratio r is finite and proportional to $(\ln M/m_\pi)^2$. The approximations here are not identical to this perturbation approximation since only the 2γ -intermediate state is included in the absorption amplitude. In order to reproduce the perturbation calculation it is necessary to include in addition the two higher mass states ($N\bar{N}$) and ($NN\gamma$) where $N(\bar{N})$ represent a (anti-)nucleon.

or $r = 2.2 \cdot 10^{-7}$, $1.2 \cdot 10^{-7}$, and $0.3 \cdot 10^{-7}$ for $p = 2M/m_\pi$, M/m_π , and 1, respectively.

It is not unreasonable to assume the possibility of detecting such a small branching ratio⁽⁶⁾. We remark only that it is necessary to observe the decay $\pi^0 \rightarrow e^+ + e^-$ with high resolution in order to reduce background from Dalitz pairs⁽⁷⁾, $\pi^0 \rightarrow \gamma + e^+ + e^-$ which occur with the very much larger branching ratio 0.0118. The three-body decay into a Dalitz pair leads to a continuous distribution of final $e^- e^+$ pair energies and angles which is strongly peaked for the pair to emerge with small relative angle and with total energy $\approx \frac{1}{2} m_\pi$ so that the one virtual intermediate photon is close to the mass shell. However, a small fraction of Dalitz pair decays will yield final photons of very small energy so that the energetic $e^+ e^-$ pair will emerge essentially back-to-back. Such events will look like direct decays $\pi^0 \rightarrow e^+ + e^-$. One can calculate directly from the papers of DALITZ⁽⁷⁾ and of KROLL and WADA⁽⁸⁾ that the branching ratio for Dalitz pairs which leave an energy of up to δm_π , with $\delta \ll 1$, on the emitted photons is $r_D = (8\alpha/3\pi)\delta^4$. For a resolution of 6% (defined by $\delta = 0.06$, or the photon energy ≤ 8.4 MeV), $r_D = 0.8 \cdot 10^{-7}$ and is less than—or of the order of—the expected values of the branching ratio r for direct $e^+ e^-$ decay.

In this connection we note that detection of an accurate spectrum of Dalitz pairs may also provide information on the structure of the vertex $\pi^0 \rightarrow \gamma + \gamma$. However, the direct decay ratio would appear to be cleaner from an experimental viewpoint and to be a more sensitive probe of high momenta and of small distances because of the momentum integration corresponding to the closed loop in Fig. 1.

In resumé then, the branching ratio, r , for $\pi^0 \rightarrow e^+ + e^-$, is a measure of pion structure as operating at the vertex of Fig. 1. (We assume validity of quantum electrodynamics.) Data on this are extremely welcome, especially in view of the present unsatisfactory state of our understanding of elementary particle structure⁽⁹⁾. It is also clear that observation of a larger branching

⁽⁶⁾ This is a much smaller branching ratio than discussed in P. LINDENFELD, A. SACHS and J. STEINBERGER: *Phys. Rev.*, **89**, 531 (1953).

⁽⁷⁾ R. H. DALITZ: *Proc. Phys. Soc. (London)*, A **64**, 667 (1951).

⁽⁸⁾ N. M. KROLL and W. WADA: *Phys. Rev.*, **98**, 1355 (1955).

⁽⁹⁾ This has been particularly emphasized recently by the work of P. FEDER-BUSH, M. L. GOLDBERGER and S. TREIMAN: *Phys. Rev.* **112**, 642 (1958). See also: J. BERNSTEIN and M. L. GOLDBERGER: *Rev. Mod. Phys.*, **30**, 465 (1958); S. D. DRELL: *VIII Annual International Conference on High-Energy Physics* (CERN, 1958, to be published). It has been discussed here that pion structure may be a clue to a better understanding of nucleon structure. The «pion structure» operating in the nucleon structure problem corresponds to pion scattering with emission or absorption of a

ratio than predicted above could be interpreted as evidence for a direct non-charge exchange interaction of the form (pion) (lepton) (lepton).

* * *

I would like to thank Dr. PETER HILLMAN for first calling my attention to this problem and for subsequent valuable discussion. I would also like to thank CERN, its Director, Professor C. J. BAKKER, and Professor B. FERRETTI, Director of the Theory Group, for their hospitality and for the award of the Ford Foundation Fellowship which made possible this visit.

virtual photon and is different from vertex (1'). A future complete analysis of (1') would relate its structure to « fundamental » ones such as the nucleon structure probed by virtual Compton scattering (4).

R I A S S U N T O (*)

Si discute e si analizza il decadimento diretto $\pi^0 \rightarrow e^+ + e^-$ quale mezzo per saggiare la « mole del pion ».

(*) Traduzione a cura della Redazione.

Introduction of a Neutral Pseudoscalar Field and a Possible Connection between Strangeness and Parity.

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(ricevuto il 9 Dicembre 1958)

Summary. — A neutral pseudoscalar field with vanishing rest mass is introduced. We first ask for the condition guaranteeing the vanishing rest mass, which turns out to be an invariance requirement under a certain kind of gauge transformation. Then assigning an appropriate transformation to each field one can restrict the possible forms of weak interactions. The main results obtained in this paper are as follows: (1) the change of strangeness necessarily requires the violation of both parity and charge conjugation; (2) the change of the number of electrons or of muons necessarily violates parity and charge conjugation; (3) strangeness can change only by unity in weak interactions.

1. — Introduction.

The τ - θ puzzle led LEE and YANG to doubt of the validity of the conservation of parity ⁽¹⁾ and various experiments that followed confirmed directly the violation of parity in weak interactions ⁽²⁻⁴⁾. The violation of parity was then attributed to the two-component neutrino ⁽⁵⁻⁷⁾ but the original τ - θ

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(¹) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956).

(²) C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPES and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957).

(³) R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957).

(⁴) J. T. FRIED and V. L. TELEGGI: *Phys. Rev.*, **105**, 1618 (1957).

(⁵) A. SALAM: *Nuovo Cimento*, **5**, 299 (1957).

(⁶) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957).

(⁷) L. LANDAU: *Nucl. Phys.*, **3**, 127 (1957).

puzzle was left behind the neutrino problem. It would, therefore, be worthwhile to ask why the parity is violated also in neutrinoless weak interactions.

In order to understand the nature of weak Fermi interactions several authors presented arguments in favor of the universal VA coupling such as two component theory⁽⁸⁾, chirality invariance⁽⁹⁾ and mass reversal invariance⁽¹⁰⁾. These arguments are, however, not fully satisfactory since, for instance, chirality invariance is valid in weak interactions only in the absence of strong interactions as well as rest mass terms, and also since mass reversal is not a quantum mechanical transformation. Consequently it seems likely that there is no room to accomodate a new invariance principle within the framework of the present theory. Hence the introduction of a new invariance principle will probably necessitate the introduction of a new degree of freedom into the theory, and we shall do so in what follows by introducing a new field. The introduction of a new field with vanishing rest mass seems to fit the present purpose, since the vanishing rest mass is generally guaranteed by an invariance principle, *e.g.* gauge invariance for the photon field and γ_5 -invariance for the neutrino field.

We have chosen a neutral pseudoscalar field denoted by B hereafter. The vanishing rest mass of this field is guaranteed by the invariance of the theory under the transformation $B \rightarrow B + \lambda$, where λ is a real constant. In order to relate this transformation to other fields, we postulate that every Fermi field is simultaneously transformed as $\psi \rightarrow \exp [ig\lambda(\gamma_5 + a)]$, where g and a are real constants chosen appropriately to each Fermion. We shall refer to the above transformation as «gauge transformation» hereafter.

In Section 2 the general rule to write down a gauge invariant Lagrangian will be given, and also the relation of the gauge transformation to the mass reversal will be discussed. In Section 3 the conservation law resulting from the gauge invariance will be given. In Section 4 we shall apply the theory to leptonic processes and determine the possible forms of Fermi interactions. In Section 5 the theory is extended to include strongly interacting particles. Assuming that all weak interactions are either of the Yukawa type or of the Fermi type and that every Fermi interaction comprises lepton wave functions, one can show that the strangeness selection rule $\Delta S = 0, \pm 1$ in weak interactions is an inevitable consequence of the theory and that the change of strangeness $\Delta S = \pm 1$ necessarily requires the violation of parity and of charge conjugation.

⁽⁸⁾ R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1957).

⁽⁹⁾ E. C. G. SUDARSHAN and R. E. MARSHAK: *Phys. Rev.*, **109**, 1860 (1958).

⁽¹⁰⁾ J. J. SAKURAI: *Nuovo Cimento*, **7**, 646 (1958). See also J. TIOMNO: *Nuovo Cimento*, **1**, 686 (1955); S. HORI and A. WAKASA: *Nuovo Cimento*, **6**, 304 (1957); K. FUJII and K. IWATA: *Progr. Theor. Phys.*, **19**, 475 (1958); **20**, 126 (1958).

2. – Introduction of a neutral pseudoscalar field.

We introduce a neutral pseudoscalar field B and ask for the condition guaranteeing the vanishing of the rest mass of the B quantum. The condition is given by the invariance of the theory under the «gauge transformation»

$$(1) \quad B \rightarrow B + \lambda ,$$

where λ is a real constant. It is worthwhile to notice that the invariance requirement under the transformation (1) with an arbitrary real function $\lambda(x)$ satisfying $\square\lambda(x) = 0$ is superfluous, since there seems to be no theory that is invariant under this generalized transformation. Furthermore the gauge transformation (1) cannot be applied to charged fields. The reason for this is that this transformation is not compatible with the electromagnetic gauge transformation of the first kind.

If the B field is coupled to other fields only through derivative couplings, the Lagrangian

$$(2) \quad L_B = -\frac{1}{2} \left(\frac{\partial B}{\partial x_\mu} \right)^2 - j_\mu \frac{\partial B}{\partial x_\mu} ,$$

is obviously gauge invariant. In order to derive useful results, however, it is necessary to assume that not only the B field but also other fields are simultaneously transformed. As a possible gauge transformation of the first kind, we take

$$(3) \quad \psi \rightarrow \exp [ig\lambda(\gamma_5 + a)] \psi ,$$

where g and a are constants appropriately assigned to this spinor field, and λ is the transformation parameter appearing in (1). It has to be noticed that the transformations (1) and (3) are given in the Heisenberg representation, and that the transformation of field operators in the interaction representation is quite different from the above ones.

Then a possible gauge invariant Lagrangian is given by

$$(4) \quad L = -\bar{\psi} (\gamma_\mu \partial_\mu + m \exp [-2ig\gamma_5 B]) \psi - \frac{1}{2} \left(\frac{\partial B}{\partial x_\mu} \right)^2 .$$

A gauge invariant interaction is always obtained by the following substitution:

$$(5) \quad m\bar{\psi}\psi \rightarrow m\bar{\psi} \exp [-2ig\gamma_5 B]\psi .$$

This general method of constructing a gauge invariant interaction corresponds to Dirac's substitution in quantum electrodynamics

$$(6) \quad \bar{\psi} \gamma_\mu \partial_\mu \psi \rightarrow \bar{\psi} \gamma_\mu (\partial_\mu - ieA_\mu) \psi .$$

In addition to the above interaction one can add interactions of the derivative coupling type as appeared in (2). The gauge invariant interactions of this kind correspond to the gauge invariant Pauli type interaction in quantum electrodynamics. It is usually taken for granted that the Pauli interaction does not exist and that the electromagnetic interaction of the Dirac type (6) is the only one realized in nature, and this postulate is called the principle of minimal electromagnetic interaction. Correspondingly we shall assume that the only interaction of the B field with other fields is given by the substitution (5) in the rest mass term. Later in Section 5 we shall see that this postulate needs a minor modification concerning the pion-baryon interactions. This is indeed an essential assumption in determining the possible forms of weak interactions. There are two points of importance: 1) The electromagnetic interactions $-ie\bar{\psi}\gamma_\mu\psi \cdot A_\mu$ is invariant under «our» gauge transformation; 2) The interaction of the B field with other fields in the absence of weak interactions conserves parity and charge conjugation separately. The B field is odd under space reflection and even under charge conjugation.

Next we shall discuss the relation of the gauge transformation to the so-called mass reversal. Under a special gauge transformation corresponding to

$$(7) \quad \lambda = \pi/2g ,$$

the rest mass term is transformed as

$$(8) \quad m\bar{\psi} \exp[-2ig\gamma_5 B]\psi \rightarrow -m\bar{\psi} \exp[-2ig\gamma_5 B]\psi ,$$

if only the B field is transformed. Since the B field appears only in the rest mass term this transformation can also be regarded as the transformation of the rest mass

$$(9) \quad m \rightarrow -m .$$

Under this discrete transformation a spinor field is transformed according to (3) as, e.g.,

$$(10) \quad \begin{cases} \psi \rightarrow i\gamma_5 \exp[i\pi a/2]\psi , \\ \psi \rightarrow \mp \gamma_5 \psi , \end{cases} \quad \text{for } a = \pm 1 .$$

The transformations (9) and (10) together form the so-called mass reversal transformation. As seen from (9) in which a c -number undergoes a transformation, mass reversal is by no means a quantum mechanical transformation, but we gave an equivalent quantum mechanical transformation in terms of a new field.

3. — Conservation law.

From the invariance of the Lagrangian (4) under the simultaneous transformations (1) and (3), one can derive a conservation law of the form

$$(11) \quad \frac{\partial J_\mu}{\partial x_\mu} = 0,$$

where J_μ is given by

$$(12) \quad J_\mu = \frac{\partial B}{\partial x_\mu} + ig\bar{\psi}\gamma_\mu(\gamma_5 + a)\psi.$$

Hence the following quantity Q is conserved:

$$(13) \quad Q = \int J_0 d^3x.$$

This conserved quantity Q corresponds to the electric charge in quantum electrodynamics. One can show that an operator of the form (12) generally leads to continuous eigenvalues. However, in our case so far as observable quantities are concerned Q has just discrete eigenvalues. This will be shown in Appendix A.

The question to what extent the above conservation law restricts the possible forms of the S matrix is not a simple subject, and we shall discuss it in a simple case in Appendix B.

4. — Application to leptons.

In this section we shall discuss the properties of leptons based upon the new invariance principle. The usefulness of this theory as applied to leptons rests fully upon the choice of a suitable set of transformations for leptons. Furthermore we shall assume the lepton conservation as well as the two component theory of neutrino.

When e^- , μ^- and two component ν are particles, we choose the following

transformations among many other possibilities:

$$(14) \quad \left\{ \begin{array}{l} \psi_\mu \rightarrow \exp [ig\lambda(\gamma_5 - 1)]\psi_\mu, \\ \psi_v \rightarrow \psi_v, \\ \text{and} \\ \psi_e \rightarrow \exp [-ig\lambda(\gamma_5 - 1)]\psi_e. \end{array} \right.$$

Then the possible Fermi interactions invariant under the above transformations are given by

1) β -decay

$$(15a) \quad G_1 \bar{\psi}_p \gamma_\sigma (1 + a_1 \gamma_5) \psi_n \cdot \bar{\psi}_e \gamma_\sigma (1 + \gamma_5) \psi_v + \text{h. c.},$$

2) μ -capture

$$(15b) \quad G_2 \bar{\psi}_p \gamma_\sigma (1 + a_2 \gamma_5) \psi_n \cdot \bar{\psi}_\mu \gamma_\sigma (1 + \gamma_5) \psi_v + \text{h. c.},$$

3) μ -e decay

$$(15c) \quad G_3 \bar{\psi}_\mu \gamma_\sigma (1 + \gamma_5) \psi_e \cdot \bar{\psi}_v \gamma_\sigma (1 + \gamma_5) \psi_v + \text{h. c.},$$

All Fermi interactions turn out to be of the VA combination just as other arguments give, and the μ -e decay gives the maximum violation of parity in this unrenormalized form. However, unlike other theories we cannot put the ratio Gamow-Teller to Fermi a_1 or a_2 equal to unity, and further there is no relationship among the three coupling constants G_1 , G_2 and G_3 . The transformation (14) is so chosen as to give negative helicity to leptons and positive helicity to antileptons in weak interactions ^(11,12). In this section baryon wave functions are treated as if they were not transformed, but we shall see in the next section that the interactions (15a) and (15b) are also invariant under the baryon transformations.

When e^- , μ^+ and four-component v are particles as proposed by KONOPINSKI and MAHMOUD ⁽¹³⁾ several years ago, the vanishing of the rest mass of the neutrino is guaranteed by the invariance of the theory under the simultaneous transformation

$$(16) \quad \psi_v \rightarrow \gamma_5 \psi_v, \quad \psi_e \rightarrow \psi_e, \quad \psi_\mu \rightarrow -\psi_\mu.$$

⁽¹¹⁾ H. FRAUENFELDER, R. BOBONE, E. VON GOELER, N. LEVINE, H. R. LEWIS, R. N. PEACOCK, A. ROSSI and G. DE PASQUALI: *Phys. Rev.*, **106**, 386 (1957).

⁽¹²⁾ M. GOLDHABER, L. GRODZINS and A. W. SUNYAR: *Phys. Rev.*, **109**, 1015 (1958).

⁽¹³⁾ E. J. KONOPINSKI and H. M. MAHMOUD: *Phys. Rev.*, **92**, 1045 (1953).

In this case we can again employ (14) provided that ψ_e as well as ψ_μ always refer to the field operators for negative leptons. As has been discussed before one cannot distinguish between these two alternative choices so far as presently existing experiments are concerned (14).

5. - Application to strongly interacting particles.

We shall now apply the requirement of gauge invariance to strongly interacting particles as it was the first intention of this paper. Among many other possibilities we shall choose the following set of transformations for these particles in analogy to the case of leptons:

1) baryons

$$(17) \quad (*) \quad \psi \rightarrow \exp \left[if\lambda \left((-1)^v \frac{\gamma_5}{2} + U \right) \right] \psi ,$$

2) mesons

$$(18) \quad \Phi \rightarrow \exp [if\lambda U] \Phi ,$$

where U is the sum of nucleon number N and strangeness S , and f is generally different from g for leptons. However, it seems likely that there is a simple relation between f and g .

If the pion-baryon interaction is of the derivative coupling type, *i.e.*

$$(19) \quad i \frac{G}{\mu} \bar{\psi} \boldsymbol{\tau} \gamma_5 \gamma_\sigma \psi \cdot \partial_\sigma \boldsymbol{\varphi} ,$$

then this interaction is already invariant under the gauge transformation (17). If, on the contrary, it is of the form of pseudoscalar coupling, *i.e.*

$$(20) \quad iG \bar{\psi} \boldsymbol{\tau} \gamma_5 \psi \cdot \boldsymbol{\varphi} ,$$

then this interaction is no longer invariant, and in order to make it invariant we follow the pattern utilized in modifying the rest mass term by substituting the following interaction for (20):

$$(21) \quad iG \bar{\psi} \gamma_5 \boldsymbol{\tau} \exp [-2if\gamma_5 B] \psi \boldsymbol{\varphi} .$$

(14) K. NISHIJIMA: *Phys. Rev.*, **108**, 907 (1957).

(*) The factor $(-1)^v$ is necessary in order to introduce leptonic decays of strange particles.

However, we need not modify the baryon-kaon interaction if it is strangeness conserving and of a non-derivative coupling type either scalar or pseudoscalar.

The conservation law that follows from the gauge invariance is given again by (11) with J modified in the following way:

$$(22) \quad J_\mu = J_\mu^B + J_\mu^U,$$

where

$$(23) \quad J_\mu^U = if \left[\sum_{\text{baryons}} U \cdot \bar{\psi} \gamma_\mu \psi + \Phi_K^* \frac{\partial \Phi_K}{\partial x_\mu} - \frac{\partial \Phi_K^*}{\partial x_\mu} \Phi_K \right],$$

$$(24) \quad J_\mu^B = \frac{\partial B}{\partial x_\mu} + \frac{1}{2} if \sum_{\text{baryons}} (-1)^r \bar{\psi} \gamma_\mu \gamma_5 \psi.$$

Correspondingly the conserved quantity Q can be split into two parts

$$(25) \quad Q = Q_B + Q_U, \quad Q_U = fU.$$

In the above expressions the contributions from leptons are discarded since they are not essential in the following discussions. In strong interactions Q_B and Q_U are conserved separately since $U = N + S$ is conserved. Now let us emphasize that they have opposite transformation properties under space reflection and charge conjugation, *i.e.*

$$(26) \quad PQ_B P^{-1} = -Q_B, \quad PUP^{-1} = -U,$$

and

$$(27) \quad CQ_B C^{-1} = Q_B, \quad CUC^{-1} = -U.$$

In weak interactions only the sum Q is conserved since U is no longer conserved, and this means that quantities of opposite transformation properties are mixed up, or in other words, that both parity and charge conjugation are violated in weak interactions. To be more precise, we shall make further assumptions. Assumption 1): Weak interactions are either of the Yukawa type or of the Fermi type. Assumption 2): Every Fermi interaction comprises lepton wave functions, or in other words, there are no Fermi interactions consisting of baryon wave functions (*).

(*) In this connection it might be suggestive to assume that lepton wave functions occur only in Fermi interactions but not in weak Yukawa interactions. Indeed Yukawa interactions of the form $K\mu\nu$, $K\bar{\nu}\nu$ cannot be gauge invariant.

From these assumptions we can draw two important conclusions:

- 1) In weak interactions the strangeness S obeys the selection rule $\Delta S = 0, \pm 1$ provided that we assign zero strangeness to leptons.
- 2) When strangeness changes by unity, one can determine the unrenormalized form of such an interaction uniquely.

In order to prove the first statement we shall show that a bilinear form consisting of baryon wave functions and changing the strangeness by two cannot be gauge invariant. Under the transformation (17) $\bar{\psi}_\Xi O \psi_N$ is transformed as

$$(28) \quad \bar{\psi}_\Xi O \psi_N \rightarrow \bar{\psi}_\Xi \exp \left[if\lambda \left(-\frac{\gamma_5}{2} + 1 \right) \right] O \exp \left[if\lambda \left(-\frac{\gamma_5}{2} + 1 \right) \right] \psi_N .$$

The invariance requires

$$\exp \left[if\lambda \left(-\frac{\gamma_5}{2} + 1 \right) \right] O \exp \left[if\lambda \left(\frac{\gamma_5}{2} + 1 \right) \right] = O ,$$

but the only possible solution of this equation is given by

$$(29) \quad O = 0 .$$

This proves the selection rule $\Delta S = 0, \pm 1$ for Yukawa interactions of the form $\bar{B}B\pi$ and for Fermi interactions under the assumption 2). This can also be proved for Yukawa interactions of the form $\bar{B}BK$. If the assumption 2) is dropped, $\Delta S = \pm 2$ cannot be excluded theoretically as given by the example $\bar{\Lambda}n \cdot \bar{\Lambda}n$.

The proof of the second statement is illustrated by considering

$$(30) \quad \Lambda^0 \rightarrow p + \pi^- .$$

The bilinear form $\bar{\psi}_p O \psi_\Lambda$ is invariant only when O assumes the form

$$(31) \quad O = \gamma_\sigma (1 + \gamma_5) ,$$

which is the only solution of the equation requiring gauge invariance

$$\exp \left[if\lambda \left(-\frac{\gamma_5}{2} + 1 \right) \right] O \exp \left[if\lambda \left(\frac{\gamma_5}{2} \right) \right] = O .$$

Hence the unrenormalized form of the decay interaction for (30) is uniquely given by

$$(32) \quad \bar{\psi}_p \gamma_\sigma (1 + \gamma_5) \psi_\Lambda \cdot \partial_\sigma \varphi .$$

Similarly for the process

$$(33) \quad \Xi^- \rightarrow \Lambda^0 + \pi^- ,$$

we get uniquely an invariant interaction of the form

$$(34) \quad \bar{\psi}_\Lambda \gamma_\sigma (1 - \gamma_5) \psi_\Xi \cdot \partial_\sigma \varphi .$$

It is very suggestive that the signs of γ_5 are different in Λ and Ξ decays showing opposite polarizations for the decay products p and Λ^0 .

Finally it must be mentioned that when strangeness does not change as in β decay the form of interaction cannot be determined uniquely as illustrated by (15a) and (15b). In this point the present theory differs from other theories basically.

To conclude the author would like to remark that as far as the coupling constants f and g are sufficiently small the B quantum would hardly be observed.

* * *

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APPENDIX A

Eigenvalue problem for the operator Q .

From the Lagrangian (4) we see that L is a periodic function of B , namely (4) is invariant under the transformation

$$(A.1) \quad B \rightarrow B + \pi/g .$$

This periodicity requires that all physically observable quantities be invariant under this transformation. In order to determine the eigenvalues of Q , we

consider an eigenvalue equation of the form

$$(A.2) \quad [Q, X_\alpha] = q_\alpha X_\alpha.$$

If the eigenoperators X form a complete set of operators, *i.e.* if the equation $[O, X_\alpha] = 0$ valid for every α requires that O be a c -number, the eigenvalues of Q are completely determined and given by

$$(A.3) \quad q = \sum c_\alpha q_\alpha, \quad (c_\alpha: \text{integer}),$$

provided that $Q\Phi_0 = 0$, where Φ_0 is the vacuum state.

For the sake of definiteness, we take $a = -1$ in (12)

$$(A.4) \quad J_\mu = \frac{\partial B}{\partial x_\mu} + ig\bar{\psi}\gamma_\mu(\gamma_5 - 1)\psi,$$

then the eigenoperators and corresponding eigenvalues are given by Table I.

TABLE I.

X_α	q_α
$(\gamma_5 - 1)\psi$	$2g$
$(\gamma_5 + 1)\psi$	0
$\psi^*(\gamma_5 - 1)$	$-2g$
$\psi^*(\gamma_5 + 1)$	0
$\exp[-2ingB]$	$2ng$
$\partial B / \partial x_\lambda$	0

The invariance under (A.1) requires that n occurring in $\exp[-2ingB]$ should be an integer so far as observable quantities are concerned, and hence we get all the eigenvalues of Q

$$(A.5) \quad q = (\text{even number}) g.$$

For this reason it is sometimes convenient to introduce $Q/2g$, since it has integer eigenvalues even after renormalization. In this connection it is desirable to choose f equal to g or a simple multiple of g .

APPENDIX B

A low energy limit theorem for the emission or absorption of a B -quantum.

Let us consider a gauge transformation U_λ which is defined by

$$(B.1) \quad U_\lambda B U_\lambda^{-1} = B + \lambda,$$

$$(B.2) \quad U_\lambda \psi U_\lambda^{-1} = \exp [ig\lambda(\gamma_5 + a)]\psi.$$

U_λ can be expressed in terms of Q by

$$(B.3) \quad U_\lambda = \exp [i\lambda Q].$$

The statement that the theory is invariant under this gauge transformation is expressed by

$$(B.4) \quad U_\lambda H U_\lambda^{-1} = H,$$

where H is the total Hamiltonian consisting of free part H_0 and interaction part H_{int} . However, this does not always lead to

$$(B.5) \quad U_\lambda S U_\lambda^{-1} = S,$$

where S is the scattering matrix. The condition for (B.5) is more severe than (B.4) and is given by

$$(B.6) \quad U_\lambda H_0 U_\lambda^{-1} = H_0, \quad U_\lambda H_{\text{int}} U_\lambda^{-1} = H_{\text{int}}.$$

(B.6) is true only for special choices of the value of λ , i.e.

$$(B.7) \quad \lambda = \frac{\pi}{g} \times (\text{integer}).$$

Let us denote $U(\pi/g)$ simply by U , then we have (B.6) and also

$$(B.8) \quad UB_{\text{in}}U^{-1} = B_{\text{in}} + \frac{\pi}{g}, \quad U\psi_{\text{in}}U^{-1} = \psi_{\text{in}}.$$

In general the transformation of incoming field operators is different from (B.1) and (B.2) and is very complicated. The simple transformation law (B.8) is valid only for special choices of the value of λ .

The S matrix is a functional of B_{in} and for this special λ we have

$$(B.9) \quad U^n S[B_{\text{in}}] U^{-n} = S \left[B_{\text{in}} + \frac{n\pi}{g} \right] = S[B_{\text{in}}].$$

Then (B.9) gives us the following result:

Let $s(k_1, k_2, \dots, p_1, p_2, \dots)$ be the S matrix element corresponding to a process in which B particles with four-momenta k_1, k_2, \dots either incoming or

outgoing as well as other particles with four-momenta p_1, p_2, \dots participate. Then (B.9) requires that s should vanish if any one of these momenta k_1, k_2, \dots is equal to zero. Thus we get a kind of low energy limit theorem. So long as S matrix elements on the energy shell are concerned we cannot expect any further relations. This result suggests that there is no difficulty corresponding to infrared divergences in quantum electrodynamics. However, the ultraviolet divergences get worse than in quantum electrodynamics as a general trend of weak interactions.

In the present paper all results are referring to unrenormalized forms of interactions. However, it should be mentioned that the conservation law can explicitly be expressed in terms of renormalized operators alone using the transformation properties of field operators.

RIASSUNTO (*)

Si introduce un campo pseudoscalare neutro con massa a riposo tendente a zero. Cerchiamo dapprima la condizione che garantisce l'annullamento della massa a riposo, e si trova essere un'esigenza d'invarianza rispetto a una certa specie di trasformazione di gauge. Assegnando poi un'appropriata trasformazione a ogni campo si può restringere il numero delle possibili forme dell'interazione debole. I principali risultati contenuti nel presente lavoro sono: 1) il cambiamento di stranezza richiede necessariamente la violazione della parità e della coniugazione della carica; 2) il cambiamento del numero di elettroni o di muoni viola necessariamente la parità e la coniugazione della carica; 3) nelle interazioni deboli la stranezza può cambiare solo di un'unità.

(*) Traduzione a cura della Redazione.

On the Spin Dependence of the Λ -N Interaction^(*)(**).

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Summary. — The ratio w^j between the lifetimes for the decays ${}^3H_\Lambda \rightarrow {}^3He + \pi^-$ and ${}^3H_\Lambda \rightarrow D + p + \pi^-$ has been calculated as a function of the spin j of the ${}^3H_\Lambda$ ($j=\frac{1}{2}$ or $j=\frac{3}{2}$). The experimental value $w_{\text{exp}} \lesssim 1$ favours $j=\frac{1}{2}$, showing that the Λ -N potential is more attractive in the singlet than in the triplet state. This result does not depend critically on the choice of the wave functions, because $w^{\frac{3}{2}}/w^{\frac{1}{2}}$ is independent of it.

1. — Introduction.

The available experimental results on the binding energies of the hyperfragments, may be interpreted by assuming that the Λ -N interaction depends on the relative spin orientation of the two particles. The phenomenological analysis⁽¹⁾ on the binding energies of the lightest hypernuclear systems allows two possible solutions for the spin dependence, which lead to two different values for the hyperfragments spins.

In this work it is shown how one can choose between these two possibilities by studying two decay processes of the ${}^3H_\Lambda$ hyperfragment, the simplest one among the known hyperfragments.

(*) This work has been presented at the XLIV Congress of the Italian Physical Society, 6-11 November 1958.

(**) When this work was completed, we have received a preprint of a paper by M. LEON, who calculates the fraction of decays in two bodies of the ${}^3H_\Lambda$ and compares it with the experimental value; his conclusions are in agreement with the results we have here obtained.

(1) R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **111**, 967 (1958).

The two decays we want to consider are:

$$(1a) \quad {}^3H_{\Lambda} \rightarrow D + p + \pi^{-},$$

$$(1b) \quad {}^3H_{\Lambda} \rightarrow {}^3He + \pi^{-}.$$

Our purpose is to calculate the lifetimes τ_a^j and τ_b^j for the two channels as functions of the spin j of the hypertriton.

It will be shown that the branching ratio $w^j = \tau_b^j/\tau_a^j$ depends strongly on the hypertriton spin j and on the ratio $x = |g_{-}|^2/|g_{+}|^2$ between the emission probabilities in s - and p -waves of the π^- meson in the free Λ decay.

The experimental value ^(2,3) $w_{\text{exp}} \lesssim 1$ singles out $j = \frac{1}{2}$, showing that the Λ -N interaction is more attractive in the singlet than in the triplet state.

This result is in good agreement with the conclusion reached by DALITZ in the study of the decay of the ${}^4H_{\Lambda}$ and with the field-theoretic calculations ^(4,5).

2. – The interaction hamiltonian for the Λ decay may be written (*) (in the non-relativistic approximation)

$$(2) \quad H' = \int d\mathbf{r} \psi_p^+(\mathbf{r}) \left[g_- + g_+ \frac{\boldsymbol{\sigma} \cdot \nabla_{\pi}}{k_0} \right] \psi_{\Lambda}(\mathbf{r}) \varphi(\mathbf{r}),$$

where

$\psi_p(\mathbf{r})$, $\psi_{\Lambda}(\mathbf{r})$, $\varphi(\mathbf{r})$ are the field operators for the proton, lambda and π respectively

$\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the three Pauli matrices,

k_0 is the π^- momentum in the free Λ decay.

If we neglect the influence of the Pauli principle in the final state for the process (1a), the lifetime τ_a^j does not depend on the spin j of the ${}^3H_{\Lambda}$ (indeed in this case the decay probability of the Λ does not depend on the orientation of its spin with respect to the deuteron spin).

⁽²⁾ For references to some ${}^3H_{\Lambda}$ decay-events see, e.g., G. FRANZINETTI and G. MORPURGO: *Suppl. Nuovo Cimento*, **2**, 785 (1957).

⁽³⁾ R. LEVI-SETTI, W. E. SLATER and V. L. TELELDI: *Suppl. Nuovo Cimento*, **10**, 68 (1958).

⁽⁴⁾ D. B. LICHTENBERG and M. ROSS: *Phys. Rev.*, **103**, 1131 (1956) and *Phys. Rev.*, **109**, 2163 (1958).

⁽⁵⁾ F. FERRARI and L. FONDA: *Nuovo Cimento*, **5**, 842 (1958).

(*) We have everywhere used units $\hbar = \text{mass } (\pi^-) = c = 1$.

The square modulus of the matrix element for the transition (1a) summed on the final spin states and averaged with respect to the initial ones, is given by

$$|\overline{H'_n}|^2 = (1 + k^2)^{-\frac{1}{2}} \left[|g_-|^2 + |g_+|^2 \frac{k^2}{k_0^2} \right] \cdot |A|^2,$$

where

$$A = \int d\mathbf{r}_n d\mathbf{r}_p d\mathbf{r}_\Lambda \Psi_D(|\mathbf{r}_n - \mathbf{r}_p|) \Phi_{^3\text{H}_\Lambda}(\mathbf{r}_n, \mathbf{r}_p, \mathbf{r}_\Lambda) \exp[-i(\mathbf{r}_\Lambda - \frac{1}{2}(\mathbf{r}_n + \mathbf{r}_p)) \cdot (\mathbf{k} + \mathbf{p})],$$

\mathbf{r}_n , \mathbf{r}_p and \mathbf{r}_Λ are the neutron, proton and Λ co-ordinates,

\mathbf{k} and \mathbf{p} are respectively the π^- and free proton momentum,

Ψ_D and $\Phi_{^3\text{H}_\Lambda}$ the space wave functions describing the deuteron and the hypertriton.

The lifetime for the process (1b) on the other hand, has a very strong dependence on j , because for $j = \frac{3}{2}$ only π^- in p -waves may be emitted in the decay.

One obtains

$$(3) \quad j = \frac{1}{2}, \quad |\overline{H'_n}|^2 = \frac{1}{12} (1 + k^2)^{-\frac{1}{2}} \left[9 |g_-|^2 + |g_+|^2 \frac{k^2}{k_0^2} \right] |A'|^2,$$

$$(4) \quad j = \frac{3}{2}, \quad |\overline{H'_n}|^2 = \frac{1}{3} (1 + k^2)^{-\frac{1}{2}} |g_+|^2 \frac{k^2}{k_0^2} |A'|^2,$$

where

$$A' = \int d\mathbf{r}_n d\mathbf{r}_p d\mathbf{r}_\Lambda \Psi_{^3\text{He}}(\mathbf{r}_n, \mathbf{r}_p, \mathbf{r}_\Lambda) \Phi_{^3\text{H}_\Lambda}(\mathbf{r}_n, \mathbf{r}_p, \mathbf{r}_\Lambda) \exp[-i\mathbf{k} \cdot \mathbf{r}_\Lambda],$$

$$\frac{k^2}{k_0^2} = 1.28,$$

and $\Psi_{^3\text{He}}$ is the space wave function for the ${}^3\text{He}$.

To calculate the explicit values of the lifetimes we shall use:

a) A Hulthén wave function to describe the final deuteron in the channel (1a)

$$\Psi_D(r) \sim \frac{1}{r} (\exp[-ar] - \exp[-br]), \quad (r = |\mathbf{r}_n - \mathbf{r}_p|),$$

where

$$(5) \quad a = 0.325, \quad b = 1.413.$$

b) Irving's (6) wave function for ${}^3\text{He}$

$$\Psi_{\text{He}} \sim \exp [-\lambda [(\mathbf{r}_n - \mathbf{r}_p)^2 + (\mathbf{r}_p - \mathbf{r}_\Lambda)^2 + (\mathbf{r}_\Lambda - \mathbf{r}_n)^2]^{\frac{1}{2}}],$$

with $\lambda = 1.3$.

The hypertriton wave function is taken to be the product of a function Φ_D , describing the relative proton and neutron positions, and a function Φ_Λ describing the relative position of the Λ and the deuteron center of mass (7)

$$(6) \quad \Phi_{^3\text{H}_\Lambda} = \Phi_D(|\mathbf{r}_n - \mathbf{r}_p|) \Phi_\Lambda(|\mathbf{r}_\Lambda - \frac{1}{2}(\mathbf{r}_n + \mathbf{r}_p)|).$$

Such a wave function disregards correlations among the positions of the three particles.

The ${}^3\text{H}_\Lambda$ is simply considered as consisting of a Λ bound to a nuclear core formed by a distorted deuteron.

This may appear an oversimplification: indeed the correlations between the positions of the Λ and the nucleons are very important if one wants to predict the strength of the interaction Λ -N by using variational methods (8).

In this case it is necessary to use a trial function for the hypertriton with sufficient flexibility in order to allow strong space correlations between the Λ and the nucleons.

For the present problem, however, the use of a simple function of the type (6) is justified by the fact that, as will be shown, the results do not depend critically upon the choice of the wave functions.

We give now to (6) the following expression

$$(7) \quad \Phi_{^3\text{H}_\Lambda} \sim \frac{1}{r} (\exp[-ar] - \exp[-br]) \exp[-\gamma \varrho], \quad (\varrho = |\mathbf{r}_\Lambda - \frac{1}{2}(\mathbf{r}_n + \mathbf{r}_p)|),$$

where a and b are given by (5).

The parameter γ has been determined in the case that the deuteron is supposed undistorted, by a variational calculation for the Λ -N interaction strength necessary to give a binding energy for the ${}^3\text{H}_\Lambda$ of 2.826 MeV.

If one takes a Λ -N interaction range which corresponds to an exchange

(6) J. IRVING: *Phil. Mag.*, **42**, 338 (1951).

(7) Various authors have used functions of this form for variational calculations on the ${}^3\text{H}_\Lambda$; see, e.g.: G. H. DERRICK: *Nuovo Cimento*, **4**, 565 (1956); J. T. JONES and J. M. KELLER: *Nuovo Cimento*, **4**, 1329 (1956); D. B. LICHTENBERG and M. ROSS: *Phys. Rev.*, **110**, 737 (1958).

(8) R. H. DALITZ and B. W. DOWNS: *Phys. Rev.*, **110**, 952 (1958).

of two π , one gets $\gamma = 0.75$; for a range corresponding to an exchange of one K-meson, one would get $\gamma = 0.67$.

By using standard methods, one can easily get the lifetimes for the two processes (1a) and (1b) and therefore the values for $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$.

The result is (for $\gamma = 0.75$):

$$(8) \quad w^{\frac{1}{2}} = \frac{0.85x + 0.407}{0.873x + 0.124}; \quad w^{\frac{3}{2}} = \frac{0.85x + 0.407}{0.496},$$

where $x = |g_-|^2/|g_+|^2$.

The available experimental data are not sufficient to determine accurately x .

They are compatible with $1 \leq x \leq 5$ (1).

In Table I, 3-rd and 4-th column, we have listed the values for $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ obtained from equations (8), for different values of x .

TABLE I. — *The numerical results for R and for the w's as functions of x. The 3rd and the 4th column refer to the wave function (7); the 5th and the 6th column refer to the wave function (9).*

x	R	$w^{\frac{1}{2}}$	$w^{\frac{3}{2}}$	$w^{\frac{1}{2}}$	$w^{\frac{3}{2}}$
1	2	1.26	2.53	1.75	3.5
1.5	2.9	1.17	3.4	1.62	4.7
2	3.77	1.13	4.25	1.55	5.84
3	5.56	1.08	6	1.48	8.2
5	9	1.04	9.4	1.41	12.7
∞	∞	0.97	∞	1.32	∞

We see that the experimental value $w_{\text{exp}} \lesssim 1$ greatly favours spin $\frac{1}{2}$ for hypertriton.

We may now ask how much this conclusion depends upon the choice for the wave functions which describe the states of ${}^3\text{H}_\Lambda$, ${}^3\text{He}$ and deuteron.

To answer this question we first study the dependence of $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ upon the parameters of the wave function (7).

In Fig. 1 we have plotted $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ against γ for $x = 1$ and $x = 2$ with a and b given by equation (5). We see from the curves that within a fairly large range of γ around 0.7 the situation is practically unchanged.

In the same way, we could analyse the dependence of the w 's upon a and b :

it can be shown (9) that in this case too, the results do not depend critically upon the choice of these parameters.

We still would like to know how the above results would be modified by a different choice of the initial wave function. We may in general expect that the values for $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ would in this case be different.

However we can easily see that the ratio

$$R = \frac{w^{\frac{3}{2}}}{w^{\frac{1}{2}}},$$

is independent of Ψ_{D} , Ψ_{He} and $\Phi_{^3\text{H}_{\Lambda}}$.

Indeed, as the lifetime for channel (1a) does not depend on the spin j , one gets:

$$R = \frac{\tau_b^{\frac{3}{2}}}{\tau_b^{\frac{1}{2}}} = \frac{|H'_{\text{fi}}^{j=\frac{1}{2}}|^2}{|H'_{\text{fi}}^{j=\frac{3}{2}}|^2} = \frac{9}{4} \frac{k_0^2}{k^2} x + \frac{1}{4},$$

where $|H'_{\text{fi}}^{j=\frac{1}{2}}|^2$ and $|H'_{\text{fi}}^{j=\frac{3}{2}}|^2$ are given by (3) and (4) respectively.

R depends only on x and when x varies between 1 and 5, R varies correspondingly between 2 and 9. This shows that even if the wave functions we have used are incorrect, the conclusion on the spin is very probably still true. Indeed to get a spin $\frac{3}{2}$ one would need to modify $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ by a factor of the order $1/R$ (in the more pessimistic case $R = 2$), which seems rather difficult to believe.

To verify this we have calculated $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ using a different wave function for ${}^3\text{H}_{\Lambda}$, namely (10)

$$(9) \quad \Phi_{^3\text{H}_{\Lambda}} \sim \frac{1}{r} (\exp[-er] - \exp[-dr]) \frac{1}{\varrho} (\exp[-\alpha\varrho] - \exp[-\beta\varrho]).$$

(9) For more details on this question see the report of our work, to be published on Rendiconti del XLIV Congresso della Società Italiana di Fisica.

(10) D. B. LICHTENBERG and M. ROSS: *Phys. Rev.*, **110**, 737 (1958). We are very much indebted to the Authors for communicating the values of the parameters which they used.

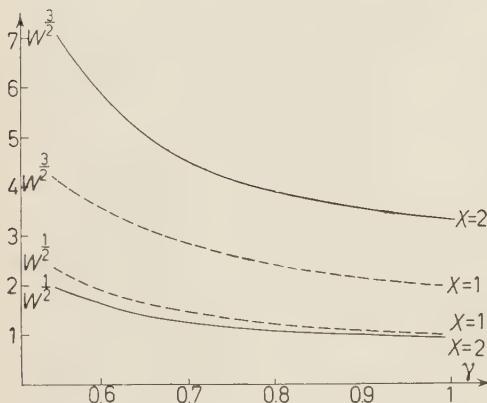


Fig. 1. — $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ obtained from (7) against γ .

The values for $w^{\frac{1}{2}}$ and $w^{\frac{3}{2}}$ one gets in this case are given in Table I, 5-th and 6-th column. We see that in this case too $w^{\frac{1}{2}}$ is much more close to the experimental value of the branching ratio.

In general we see from Table I that the agreement improves with increasing x , which could perhaps be taken as an indication that x is indeed fairly larger than 1. However this cannot be considered as a definite result due to the approximations involved in the calculations and the uncertainty of w_{exp} .

In conclusion we may say that our results are in favour of spin $\frac{1}{2}$ for ${}^3\text{H}_{\Lambda}$ and therefore for a stronger attraction in the singlet than in the triplet state of the Λ -N system.

* * *

The authors wish to express their thanks to Prof. L. A. RADICATI for his constant interest in this work and for many helpful and stimulating discussions, and to Prof. D. B. LICHTENBERG for a valuable conversation.

RIASSUNTO

Si calcola in funzione dello spin j dell' ${}^3\text{H}_{\Lambda}$ ($j = \frac{1}{2}$ o $j = \frac{3}{2}$) il rapporto w^j fra le vite medie dei due decadimenti ${}^3\text{H}_{\Lambda} \rightarrow {}^3\text{He} + \pi^-$ e ${}^3\text{H}_{\Lambda} \rightarrow \text{D} + \text{p} + \pi^-$. Il miglior accordo col valore sperimentale $w_{\text{exp}} \lesssim 1$ si ha per $j = \frac{1}{2}$, e ciò mostra che il potenziale Λ -N è maggiormente attivo nello stato di singoletto che in quello di tripletto. Questo risultato non dipende criticamente dalla scelta delle funzioni, poichè $w^{\frac{3}{2}}/w^{\frac{1}{2}}$ ne è indipendente.

NOTE TECNICHE

Camera a bolle a propano.

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Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 4 Dicembre 1958)

Riassunto. — Si descrive una camera a bolle a propano di circa due litri di volume e si indicano scopo ed alcune modalità dell'impiego fattone al sincroscicolotrone da 600 MeV del CERN. Il meccanismo di espansione, a membrana, permette una ricompressione rapida: la proporzione di tempo sensibile può quindi essere notevole. Sistema di riscaldamento assai pronto e flessibile e particolare struttura delle finestre determinano, senza inconvenienti di altra natura, buone caratteristiche termiche. Un condensatore di luce conferisce maggiore efficienza al sistema usato di illuminazione a campo oscuro.

1. — Introduzione.

La struttura generale dello strumento, in vetro e metallo con guarnizioni, ha costituito il naturale sviluppo di precedente attività di questo Laboratorio in materia ⁽¹⁾. Sia per renderne praticamente certo il regolare funzionamento ed eliminare quasi del tutto il rischio legato all'impiego di fluido infiammabile, sia per ridurre le distorsioni così reali come ottiche ⁽²⁾ ed ottenere un'illuminazione più intensa a parità di energia dissipata nel flash, sono stati però impiegati alcuni nuovi accorgimenti la cui conoscenza si ritiene possa essere utile per chi si trovi a dover impiegare camere a bolle, particolarmente a propano, in analoghe circostanze: essi risultano, coi vantaggi che ne derivano

⁽¹⁾ P. BASSI, A. LORIA, J. A. MEYER, P. MITTNER e I. SCOTONI: *Nuovo Cimento*, **4**, 491 (1956).

⁽²⁾ P. BASSI, R. CANO, S. FOCARDI, A. MICHELINI e F. SAPORETTI: *Report on «Distortion of tracks in bubble chambers»*, presented to the 2nd International Conference «Atoms for Peace» in Geneva (to be published).

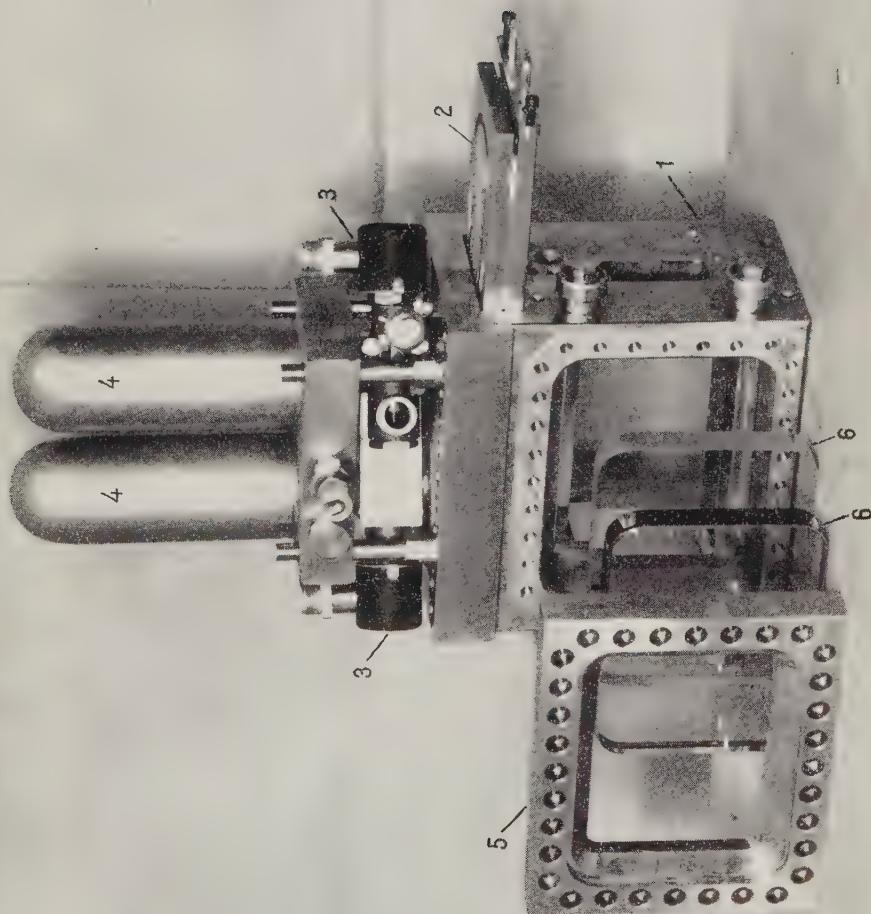
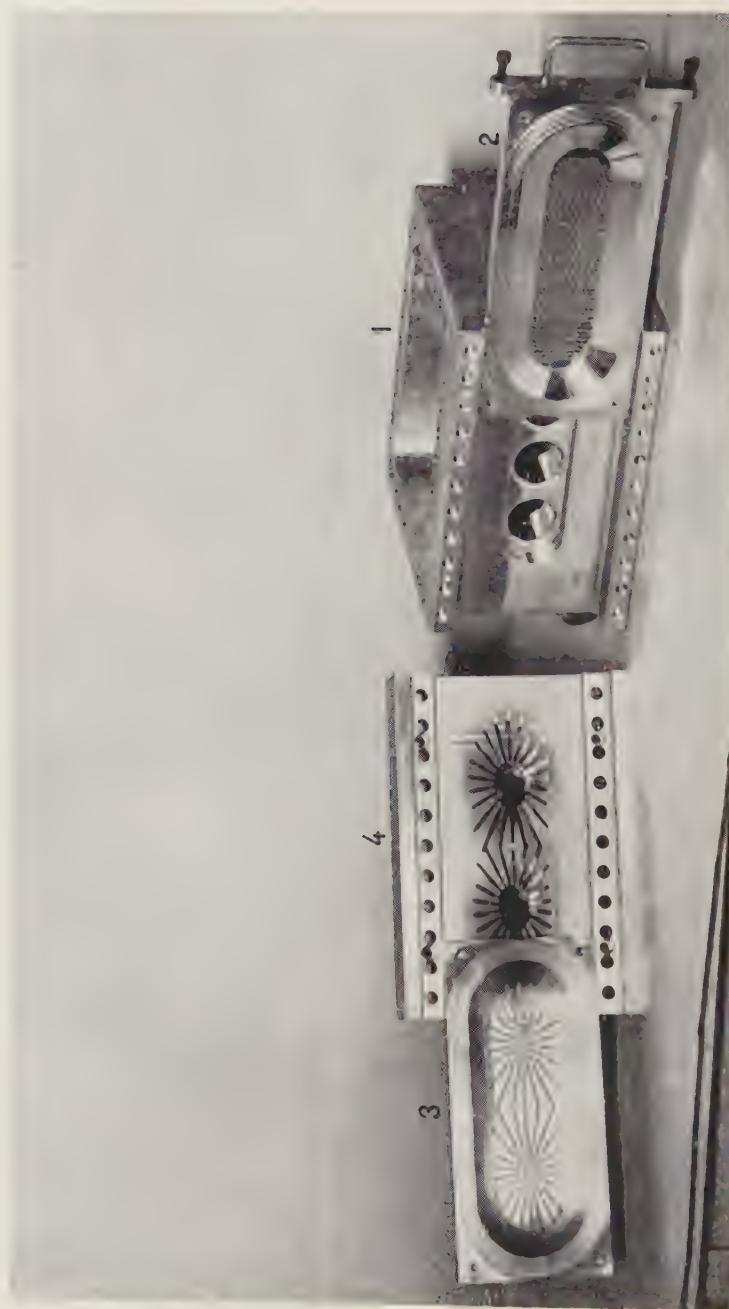


Fig. 1.



Vita

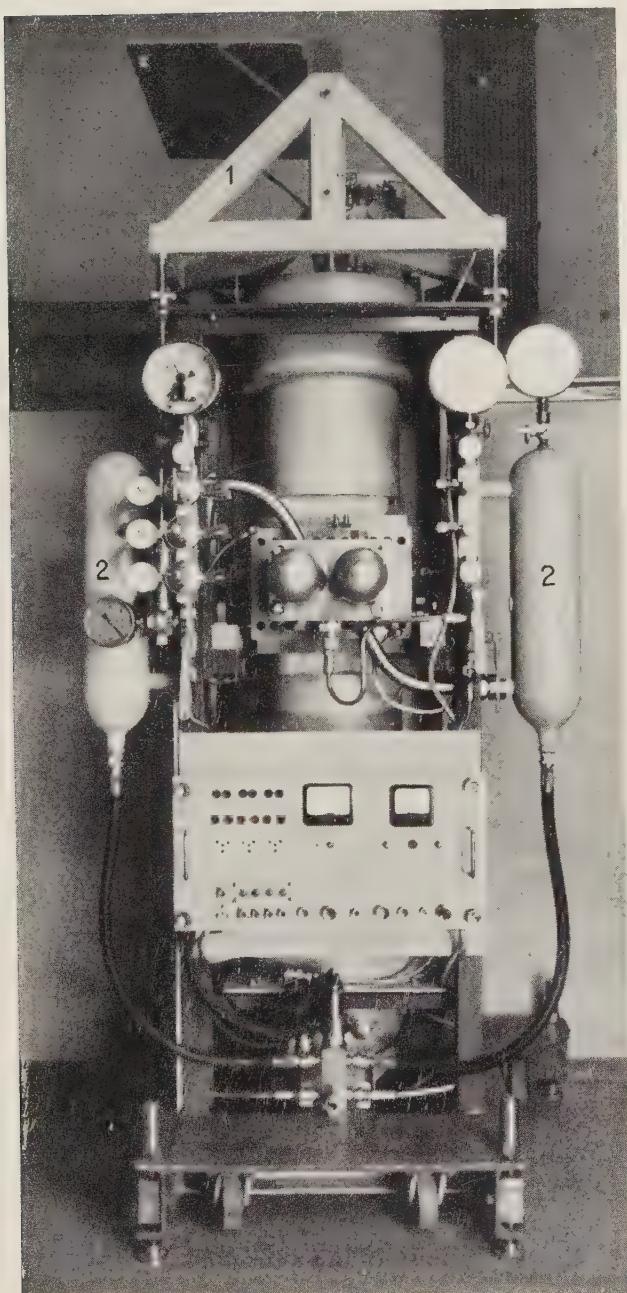


Fig. 3.

dalla dettagliata descrizione che segue, ma si ritiene opportuno segnalare particolarmente:

- 1) la struttura composta delle finestre (Fig. 1);
- 2) il sistema di riscaldamento a sorgenti indipendenti situate nel blocco metallico;
- 3) il meccanismo rapido di espansione, a membrana mobile fra due diaframmi metallici (Fig. 2);
- 4) il manometro a ponte, preciso e sensibile, per seguire dal locale di comando e controllo l'andamento della pressione nel propano;
- 5) lo specchio sferico condensatore usato nel sistema di illuminazione.

La Fig. 3 in cui la camera appare montata sul suo telaio con dispositivi sussidiari, fra cui la macchina fotografica stereoscopica, mostra come sia stata realizzata una struttura compatta, assai robusta, facilmente trasportabile e mobile.

Lo strumento è stato costruito e provato nell'Istituto di Fisica dell'Università di Padova per essere impiegato al sincrociclotrone da 600 MeV del CERN. Nella prima esposizione, compiuta di recente, non si è verificato alcun inconveniente degno di rilievo. In quasi otto ore di funzionamento continuato sono state fatte circa sedicimila fotografie al beam di π^+ da 130 MeV; la loro qualità è adeguata allo svolgimento del previsto programma di misure e calcoli diretto a determinare la sezione d'urto differenziale nello scattering elastico del pion su protone.

2. – Struttura e funzionamento.

Il volume utile, di dimensioni $7 \times 14 \times 21$ cm, è ricavato da un blocco di ottone massiccio (Fig. 4). Le facce 14×21 cm sono in realtà due grandi finestre ognuna delle quali è chiusa con due lastre di vetro temprato spesse 30 mm: fra esse è situato un foglio di plexiglas di spessore 4 mm, destinato a proteggere la lastra esterna dal diretto urto di schegge nel caso, molto improbabile, che quella interna cedesse alla pressione. Il foglio di plexiglas è immerso in olio mantenuto a pressione metà della massima raggiunta dal propano, cosicché lo sforzo viene diviso fra le due lastre.

Questa soluzione ha permesso di rinunciare ad un involucro esterno, dapprima preso in considerazione per raggiungere il grado di sicurezza ritenuto necessario. Poiché l'insieme delle lastre ed intercapedine offre grande resistenza alla propagazione del calore, anche le caratteristiche termiche sono soddisfacenti: la continuità ottica è, d'altra parte, assicurata. Il regime di temperatura viene raggiunto e facilmente mantenuto con buona stabilità, mediante quattro resistenze elettriche indipendenti collocate in apposite cavità nel corpo metallico.

Una delle facce 7×21 presenta tre grandi fori di diametro 45 mm, esternamente ai quali è sistemata la membrana formata (nell'ordine) da un foglio di mylar di 0.1 mm di spessore e da uno di gomma di 5 mm. La membrana è situata fra due diaframmi di ottone fittamente forati che le consentono una corsa massima di circa 2 cm: di essi uno è sempre immerso in propano, a

distanza tuttavia notevole dalla zona illuminata, e l'altro in aria. Il complesso formato dalle lastre forate è facilmente estraibile a mo' di cassetto, il che rende semplice e facile il cambio della membrana. Contrariamente a quanto si temeva, il caratteristico ciclo termodinamico dello strumento non è stato affatto turbato dalla presenza del diaframma nel propano, che costituisce una preziosa barriera rigida di sicurezza.

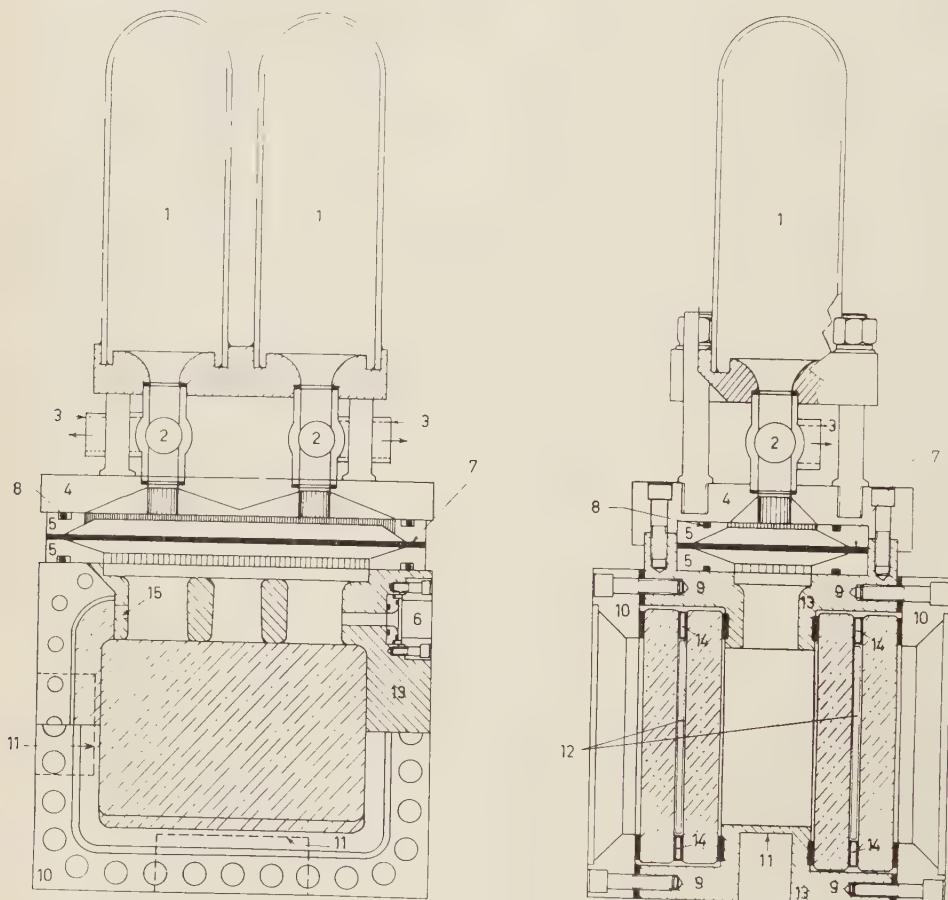


Fig. 4.

Il meccanismo di espansione è tanto rapido che, se non entrassero in gioco altre limitazioni, potrebbe compiere fino a 30 cicli al secondo: la grande superficie della membrana favorisce poi certamente la formazione di onde di pressione pressoché piane nella camera, requisito necessario per avere distorsioni piccole. Al di là della membrana, oltre un volume funzionale che costituisce il retrocamera, si trovano, installate su una flangia terminale, due valvole elettromagnetiche a tre vie, disposte in parallelo: esse aprono il retrocamera o verso due serbatoi ad alta pressione (fase di compressione) o verso altri due a bassa pressione (fase di espansione, contro pressione).

Sulla faccia 7×21 cm opposta al meccanismo di espansione è ricavata esternamente una nicchia destinata all'ingresso del beam: una nicchia analoga si trova su una delle due facce 7×14 cm, cosicché vi è possibilità di scelta per il percorso delle particelle nella camera.

I fili di una coppia termoelettrica raggiungono il propano attraverso un foro nel blocco d'ottone. Un altro foro passante di diametro 10 mm è chiuso da una lastrina di acciaio le cui deformazioni elastiche, con ottima approssimazione proporzionali alla differenza di pressione sulle due facce, si trasmettono agli anodi mobili di due triodi RCA 5734. Questi (Fig. 5) sono stati montati su due lati adiacenti di un ponte di Wheatstone e disposti in modo che i segnali di placca dovuti alla medesima deformazione della lastrina diano origine sulla diagonale di uscita ad un segnale composto di intensità pressocchè doppia, mentre le derive di caratteristica e di accensione dei triodi si compensano. Il dispositivo, la cui frequenza propria è circa 10 000 Hz, è idoneo a funzionare, nel nostro caso, per pressioni così statiche come variabili. Equilibrando il ponte sul valor massimo della pressione, si hanno 80 V in uscita per il valor minimo: il segnale era quindi di intensità sufficiente per essere inviato direttamente all'oscillografo nella stanza di comando e controllo, situata a circa 100 m di distanza.

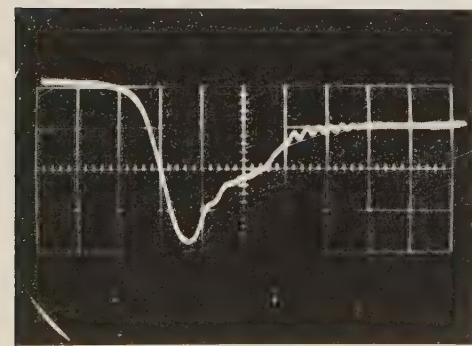


Fig. 6.

20 atm. In relazione a ciò i serbatoi ad alta pressione vengono mantenuti da un compressore a 25 atm, quelli a bassa e l'olio fra le lastre di vetro a 10 atm: regolazioni e stabilizzazioni così delle pressioni come delle temperature sono affidate a strumenti d'uso corrente.

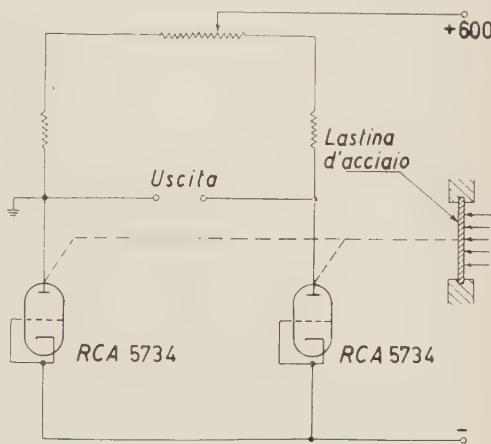


Fig. 5.

Si è rivelato inoltre particolarmente utile controllare spesso, durante l'esposizione, il funzionamento della camera con una macchina fotografica « Polaroid » che fornisce in un minuto positive sviluppate.

L'inizio della sensibilità del propano si manifesta verso i 56°C con la produzione di bollicine isolate all'atto dell'espansione in presenza di radio; il funzionamento diviene soddisfacente fra 58°C e 60°C , intervallo di temperatura cui corrisponde la tensione di vapor saturo di circa

Le valvole elettromagnetiche sono azionate scaricando, a mezzo di un thyratron, un condensatore attraverso le due bobine in parallelo (induttanza complessiva 0.5 H). La scarica cessa dopo un tempo eguale a metà del periodo proprio del circuito oscillante; variando la capacità del condensatore si regola la durata dell'impulso di corrente: la tensione del condensatore ne determina ovviamente l'ampiezza.

La Fig. 6 è una fotografia dell'oscillografo azionato come sopra descritto, in condizioni normali di lavoro. La pressione del propano comincia a scendere 13 ms dopo l'accensione del thyratron: in circa 5 ms essa raggiunge il valore minimo di 12 atm e quindi riprende a salire per raggiungere la pressione di vapor saturo. Il ciclo può ritenersi concluso a 30 ms dall'inizio: il tempo sensibile è di circa 5 ms.

Nell'esposizione al sincrociclotrone del CERN il funzionamento della camera veniva sincronizzato con quello del sincrociclotrone in modo che un impulso del beam ogni sessantaquattro trovava il propano a pressione minima: il flash seguiva il beam con un ritardo di $(0.6 \div 0.8)$ ms. Gli impulsi erano 55 al secondo: le espansioni si succedevano dunque a circa 1.1 s di intervallo.

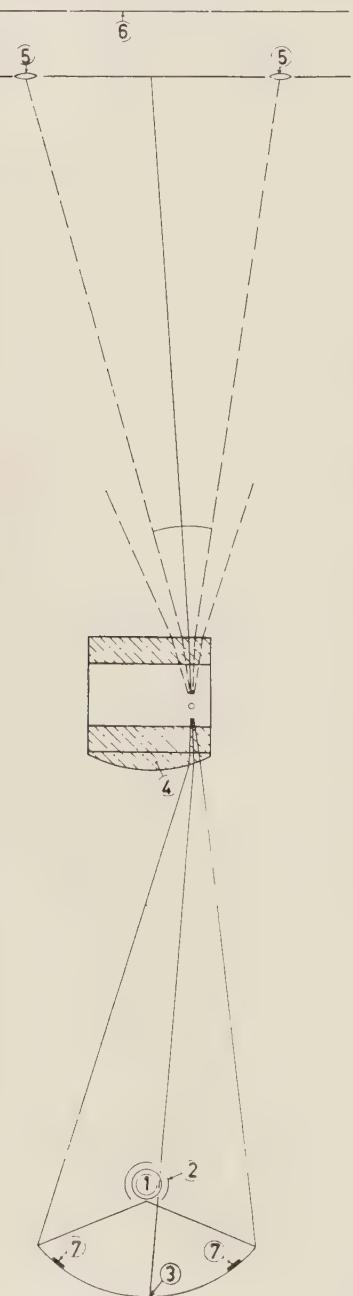


Fig. 7.

3. — Illuminazione e fotografia.

Il sistema ottico è schematicamente descritto in Fig. 7. La camera è illuminata da un flash PTW, racchiuso in uno schermo opaco, bianco diffondente nella parete interna e dotato, dalla parte opposta alla camera, di una finestra rettangolare di forma simile alla camera stessa. Uno specchio sferico agendo da condensatore forma un'immagine reale della finestra, di dimensioni uguali a quelle della camera, proprio nell'interno di questa.

Una lente sferica a contatto ottico con la camera forma invece un'immagine reale dello specchio sul piano degli obiettivi: i raggi uscenti dal vertice dello specchio passano quindi per il punto di mezzo del segmento individuato dai due obiettivi.

A. LORIA, P. MITTNER, I. SCOTONI e G. ZAGO

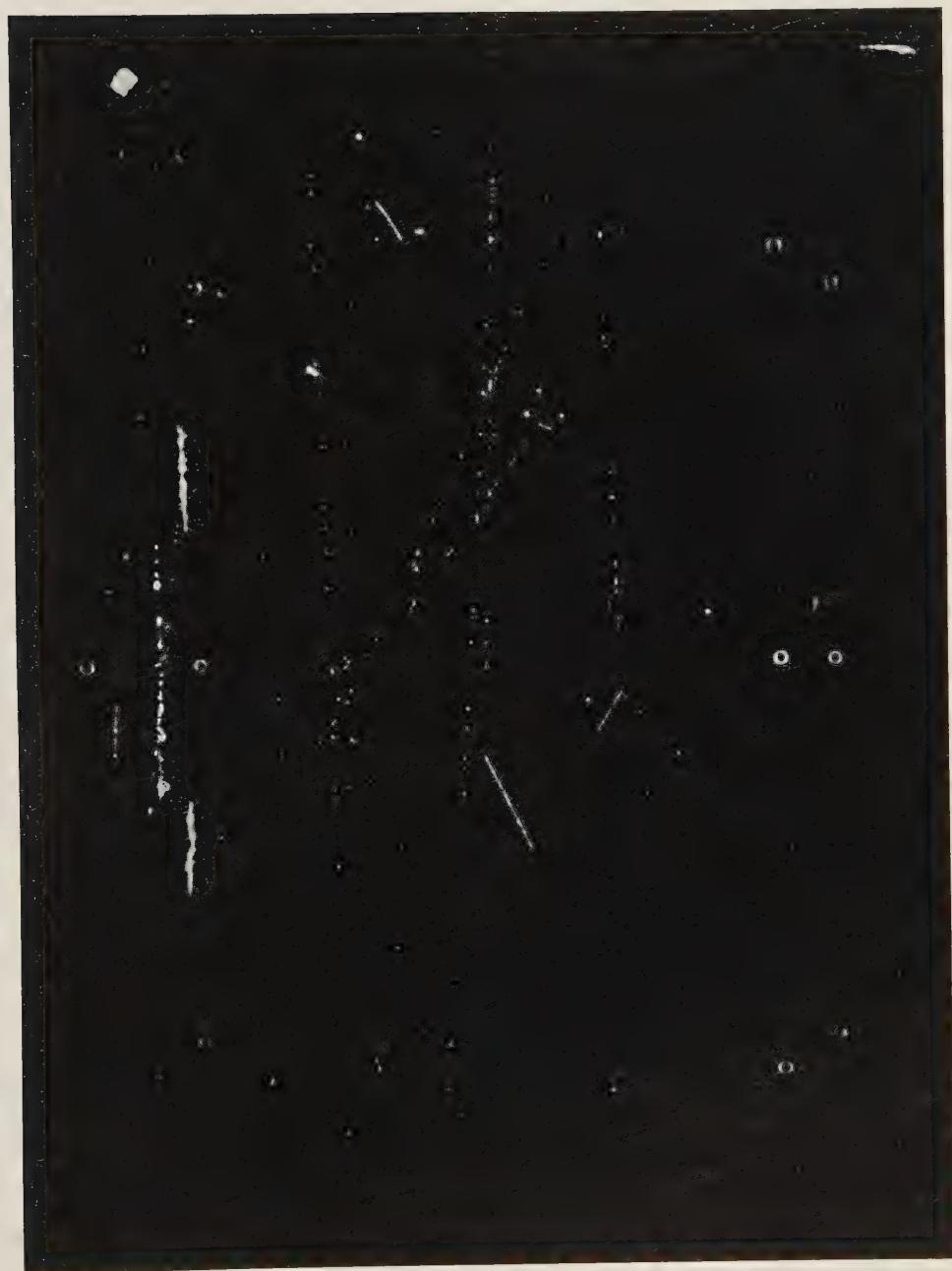


Fig. 8.

Tali raggi sono assiali per i fasci di luce che (per effetto dello specchio) convergono sui punti della camera a bolle: si riconosce che in tal modo si ottiene la massima e più uniforme illuminazione media contemporanea degli obiettivi da parte di ogni centro diffondente luce dall'interno della camera. Per evitare che gli obiettivi vengano anche direttamente illuminati dallo specchio basta annerire le due regioni (piccole) ove, sullo specchio, si formano le immagini dei due obiettivi date dalla lente.

I due obiettivi sono Schneider-Angulon $f = 65$ mm, usati con apertura 1/16. I due fotogrammi vengono presi su di un unico film da 35 mm: l'ingrandimento è circa 1/9 e l'angolo stereoscopico 25° .

Usando film pancromatico di sensibilità 17/10 din si ha luce sufficiente con 2 J per flash. Uniformità di illuminazione e contrasto possono essere giudicati dalla Fig. 8.

4. - Distorsioni.

Una valutazione delle distorsioni è stata fatta su tracce di pioni di 230 MeV con lo stesso metodo delle differenze seconde usato nelle misure di scattering in lastre nucleari.

Nella seguente tabella sono riportati i risultati delle misure relative a celle di lunghezza t eguale rispettivamente a 0.92 cm, 1.84 cm e 4.6 cm. Il Δ''_{noise} rappresenta il disallineamento medio di tre punti, del resto qualsiasi, che si succedono a intervallo t sulla traccia e più precisamente, se si segue l'ordine di osservazione, la distanza media del terzo punto dalla retta dei primi due; ciò evidentemente solo per tracce il cui scattering coulombiano è trascurabile.

t (cm)	Δ''_{tot}	Δ''_{scatt}	Δ''_{noise}
0.92	49 ± 3	20	35 ± 3.5
1.84	75 ± 10	59	45 ± 7
4.60	190 ± 40	245	—

Il risultato relativo a celle di 4.6 cm è privo di significato: in ogni modo è evidente il fatto che il Δ''_{noise} cresce con la lunghezza della cella molto più lentamente che il Δ''_{scatt} . Si è inoltre trovato che le distribuzioni di Δ''_{tot} sono, entro gli errori statistici, gaussiane con valore medio zero: la camera non appare quindi affetta da apprezzabili distorsioni sistematiche.

* * *

Ci è gradito ringraziare il Dr. R. SANTANGELO che ha collaborato alla messa a punto, misura delle distorsioni, ed impiego dello strumento. Validissima è stata per la sua costruzione l'opera dei tecnici BRUNO BALLIN, ADELMO DOZZA e GASTONE RIZZATO; preziosa, per l'elettronica, l'assistenza del perito industriale BRUNO TIVERON.

S U M M A R Y

A propane bubble chamber of about two liters volume is described: details concerning the membrane expansion mechanism, the structure of the windows and the illuminating system are given. Some features of the use of it, recently made at the CERN sinrocyclotron, are indicated.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Possible Polarization Measurements in K^+H Scattering (*).

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(ricevuto l'11 Gennaio 1959)

According to present data, the large drop in the elastic K^+H differential scattering cross-section that is apparently observed in the center-of-mass angular interval $-\frac{2}{3} > \cos \theta > -1$ is rather difficult to explain in terms of a small number of phase shifts (¹). It is stated that the data can be fitted by mainly S wave scattering except in this latter backward scattering interval which appears to be somewhat of an anomaly. Hence, additional measurements in the region would be most welcome.

Although there is some question of immediate applicability at present intensities, it is perhaps well to point out that further information about the K^+H interaction particularly its spin dependent nature can be obtained from a measurement of the possible polarization of the recoil proton.

Assuming a spin zero K^+ -meson and a spin $\frac{1}{2}$ proton, the scattering amplitude takes the comparatively simple form

$$A(\theta) + B(\theta)\boldsymbol{\sigma} \cdot \mathbf{n},$$

where \mathbf{n} is a unit vector perpendicular to the plane of scattering, $\boldsymbol{\sigma}$ the proton

(*) Supported in part by the U. S. Atomic Energy Commission.

(¹) J. E. LANNUTTI, G. GOLDHABER, S. GOLDHABER, W. W. CHUPP, S. GIAMBUZZI, C. MARCHI, G. QUARENTE and A. WATAGHIN: *Phys. Rev.*, **109**, 2121 (1958).

Pauli spin matrix, and $A(\theta)$ and $B(\theta)$ well known functions of the center-of-mass scattering angle θ and the phase shifts. The expression for the differential cross-section and polarization as functions of $A(\theta)$ and $B(\theta)$ are now equally well known and conveniently obtained (²).

With the present techniques, a determination of the proton polarization generally involves a measurement of the asymmetry resulting from a second elastic or quasi-elastic scattering of the proton at relatively high proton energies (³). Presumably, with the advent of sufficiently high K^+ intensities such measurements could be carried out in emulsions (⁴⁻⁷) and bubble chambers as well

(²) See, for example, K. B. MATHER and P. SWANN: *Nuclear scattering* (Cambridge, 1958), p. 269 *et seq.*

(³) Experimental references are listed in ref. (²) and in L. WOLFENSTEIN: *Ann. Rev. of Nucl. Sci.*, **6**, 43 (1956).

(⁴) J. FRIEDMAN: *Phys. Rev.*, **104**, 794 (1956).

(⁵) B. J. FELD and B. C. MAGLIC: *Phys. Rev. Lett.*, **1**, 375 (1958).

(⁶) G. J. BATTY and S. J. GOIDSACK: *Proc. Phys. Soc.*, **72**, 1130 (1958).

(⁷) In emulsion, it is hoped that once the usual K^+H scattering event is located and measured, it would require relatively little additional effort to follow the recoil proton track for a nuclear scattering in spite of the relatively long mean free path of the proton.

as counters (3). However, first we should note that, kinematically, obtaining recoil protons of such energies would be entirely feasible.

A simple calculation (8) shows that since the rest mass of the K^+ meson is somewhat larger than half the proton rest mass, for backward scattering of the K^+ , *i.e.* forward scattering of the recoil proton, better than 90% of the K^+ kinetic energy is transferred to the recoil proton even at low energies. In fact, at a recoil scattering angle of 25° , better than 75% of the K^+ energy is transferred to the recoil proton. Therefore, at recoil angles less than 25° , for example, a 200 MeV K^+ yields a recoil proton whose

kinetic energy is larger than 150 MeV. Such energies are ample for the usual asymmetry measurements. Finally, we note that the center-of-mass scattering angle θ is related to the proton recoil angle Θ by the simple relation (non-relativistic)

$$\theta = \pi - 2\Theta .$$

Thus it is readily seen that kinematically, the region of large energy transfer includes the region of interest, where $-\frac{2}{3} > \cos \theta > -1$ mentioned above (9).

(8) For formulae containing the non-relativistic and relativistic kinematics of recoil scattering, see for example, ref. (2), Appendix A.

(9) Similar statements could also be made about the recoil proton obtained from K^+ -neutron charge exchange scattering. This case differs from K^+ -H scattering in that it includes the effects of both the isotopic spin zero and one amplitudes whereas K^+ -H involves only the pure isotopic spin one states.

Determination of the Mass of the Λ^0 Hyperon.

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Institute for Nuclear Research - Warsaw

(ricevuto il 22 Gennaio 1959)

A stack of 180 pellicles $10 \text{ cm} \times 10 \text{ cm}$ of $600 \mu\text{m}$ Ilford G-5 emulsion was exposed to the Berkeley K^- beam. In the 80 plates of the stack which were at our disposal a search was made for decays of Λ^0 hyperons. The region in which the majority of the K^- mesons stopped was area scanned for σ_{K^-} - and σ_{π^-} stars. The pion tracks from the σ_{π^-} -stars were traced back to their point of origin: σ_{K^-} stars, two-prong stars, scattered pions from the beam, etc. If no unambiguous origin was reached in a distance of 20 mm from the σ_{π^-} star, the track was abandoned. Sixty one two-prong stars with a second outgoing track were found out of 660 pions that were followed in this way. In three cases the second track left the stack. The remaining 58 with measurable ranges were included in the analysis.

1 - Calibration

From an additional scanning performed on σ_{K^-} stars, 22 protonic decays of the Σ^+ hyperons at rest were found. Their analysis, using the method devel-

oped by FRY and WHITE⁽¹⁾, yielded for the shrinkage factor of our emulsion the value:

$$s_1 = 2.20 \pm 0.033 ,$$

the value being referred to standard conditions in our laboratory, and for the mean range of the decay protons the value:

$$R = (1676.2 \pm 7.3) \mu\text{m} .$$

Independent information on the shrinkage factor was obtained from an analysis of 53 two-prong stars classified as good examples of the Λ^0 hyperon decays (see below).

Taking into account the random errors of the individual Q_{Λ^0} values and also their known dependence on the shrinkage factor, we found for the most probable value of the shrinkage factor:

$$s_2 = 2.23 \pm 0.045 .$$

⁽¹⁾ W. F. FRY and G. R. WHITE: *Phys. Rev.*, **90**, 207 (1953).

The weighted mean of the two independent estimates s_1 and s_2

$$s = 2.21 \pm 0.027$$

was used in further calculations.

The density of our emulsion as communicated by Dr. E. J. LOFGREN was contained between 3.809 g/cm^3 and 3.853 g/cm^3 . Using this information we estimated the ratio of ranges in standard emulsion (⁽²⁾, p. 10) to those in our emulsion:

$$R_{st}/R = 1.0033 \pm 0.0045.$$

The ratio R_{st}/R was evaluated independently using our value of the mean range of the Σ^+ decay protons (^(*)) and the corresponding value of this range calculated for standard emulsion from the mass value of the Σ^+ hyperon from reference (⁽³⁾). Calculation yielded:

$$R_{st}/R = 0.9993 \pm 0.0055.$$

The weighted mean value of these two independent estimates of the R_{st}/R ratio:

$$R_{st}/R = 1.0017 \pm 0.0035$$

was used in further calculations.

2 - Measurements and errors

As already mentioned, 58 two-prong stars tentatively identified as $\Lambda^0 \rightarrow p + \pi^-$ decays were chosen for detailed analysis. Ranges of the decay products were measured to an accuracy of 1%. Each measurement was made independently by two observers. A Koristka MS2 microscope coupled to a Zeiss Optimeter was used for depth measurements. Dur-

ing each series of measurements, previously selected standards of depth in the emulsion were measured to allow conversion of the data to our standard conditions.

In evaluating random errors of the Q -values, errors arising from straggling and multiple scattering were compounded with errors of measurements of the ranges and of projected and dip angles, and also with those connected with humidity fluctuations.

Systematic errors were split up into those resulting from the uncertainty in the shrinkage factor, the uncertainty in the stopping power of our emulsion, the uncertainty in the range-energy relation, and the uncertainty in the values of the rest mass of protons and pions.

The values of straggling were taken from the work of Barkas and Young (⁽⁴⁾); uncertainty of 0.5% in the Barkas range-energy relation (⁽⁵⁾) was assumed (⁽⁶⁾); the mass values used (⁽³⁾) were

$$m_p = (938.213 \pm 0.01) \text{ MeV},$$

$$m_{\pi^-} = (139.63 \pm 0.06) \text{ MeV}.$$

3 - Results

The histogram in Fig. 1 shows the distribution of the Q -values as calculated for the 58 two-prong stars. Fifty four



Fig. 1. — Histogram of the Q -values for 58 two-prong stars with both outgoing particles stopping in the emulsion. Q -values were calculated assuming $\Lambda^0 \rightarrow p + \pi^-$ decay.

(²) W. H. BARKAS *et al.*: UCRL-3768 (1957).

(^{*}) The mean range $R = (1679.2 \pm 7.3) \mu\text{m}$ was obtained by using the weighted value of s .

(³) W. H. BARKAS and A. H. ROSENFELD: UCRL-8030 (1958), VI-41.

(⁴) W. H. BARKAS and D. M. YOUNG: UCRL-2579 (1954).

(⁵) W. H. BARKAS: UCRL-3769 (1957).

(⁶) W. H. BARKAS *et al.*: Padua-Venice Conference (1957), VI-41.

of them are concentrated in the vicinity of the mean value; in one of them the π^- -meson is inelastically scattered and its energy was determined by ionization measurements; therefore the Q -value for this event has a rather large error ($Q = (39 \pm 7)$ MeV). The remaining four events are well out of this grouping; of these four events, one ($Q = 41$ MeV) is situated 2 mm from the edge of the emulsion in a strongly distorted region, where measurements are not reliable. Three to five events of our sample thus appear to be background. Fifty three

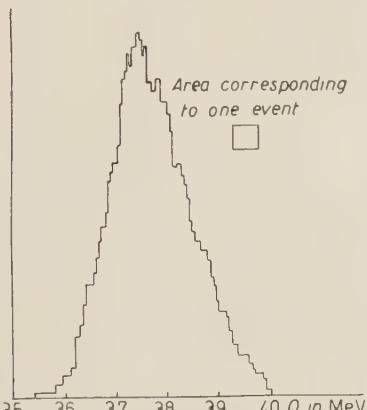


Fig. 2. — Constant area histogram for Q -values of 53 two-prong stars (see text).

events yielding closely concentrated Q -values were analysed in greater detail. A constant area histogram of the 53 Q -values where the random errors were taken into account is shown in Fig. 2.

The weighted mean value of the Q distribution is

$$Q = (37.58 \pm 0.18) \text{ MeV},$$

in good agreement with the Barkas value (6)

$$Q = (37.45 \pm 0.17) \text{ MeV}.$$

The error quoted includes both random and systematic errors. The standard deviation of the mean of the distribution due to random errors is 0.082 MeV, in good agreement with the standard deviation of the mean, calculated from the spread of the observed Q -values — 0.084 MeV.

For the mass of the Λ^0 hyperon we obtain:

$$m = (1115.42 \pm 0.19) \text{ MeV}.$$

* * *

We are greatly indebted to Dr. E. SEGRÈ for making the stack available, to Dr. E. J. LOFGREN for exposure of the stack and to Dr. S. GOLDHABER for processing the emulsion. We wish to thank Dr. P. ZIELIŃSKI for many valuable discussions. We also would like to express our appreciation to the scanning staff of our laboratory: Mrs. K. BOBIŃSKA, Mr. R. DĄBROWSKI, Mrs. M. PAZDANOWSKA, Miss W. SANIEWSKA for their careful work and especially Mrs. I. PRZYPKOWSKA for her efficient help in scanning, measurements and calculations.

**Hyperfragments from the Ξ Particle
and the Determination of the Ξ Parity.**

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(ricevuto il 23 Gennaio 1959)

The energy now available in accelerators is sufficient to produce the Ξ particle. Hence the amount of data concerning the Ξ particle should considerably increase in the nearest future. It is possible that bound systems of Ξ with nucleons may be observed. The Ξ particle in the vicinity of nucleons enters into the fast process $\Xi + N \rightarrow 2\Lambda + 30$ MeV. Therefore only those systems can exist in which the above reaction is forbidden. This forbiddness may have two reasons: Large binding energy (e.g. ${}^5\text{He}_{\Xi}$, for $B_{\Xi} > 9.5$ MeV, is stable)⁽¹⁾, and the conservation of total isotopic spin of the system and its projection. A consequence of this is that $2p\Xi^0$, $2n\Xi^-\Xi^+N$ and other hyperfragments should be stable with respect to fast transitions into 2Λ . If we neglect electromagnetic interactions the Ξ bound with a nucleon in the $T=1$ state can decay only in cascade. If one takes into account electromagnetic interactions, then the states $T_0 = \pm 1$ ($\Xi^0 p$, $\Xi^- n$) too can decay only by the cascade process $\Xi^0 p \rightarrow \pi^0 + \Lambda + p \rightarrow \pi^0 + p + (\pi^- + p)$.

On the other hand, the state $T=1$, $T_3=0$, i.e. $(1/\sqrt{2})(\Xi^- p - \Xi^0 n)$, as a result of the Coulomb energy between the Ξ^- and p , becomes a superposition of the states $T=1$ and $T=0$ (the coefficient for $T=0$ is proportional to e^2) and therefore the decay into 2Λ can take place only if there are no additional effects which forbid it, but the lifetime will be $1/e^4$ longer. In this case the decay via $2\Lambda + \gamma$ will be a competitive process. We have here a certain similarity to the case of the Σ^0 .

For the $(1/\sqrt{2})[\Xi^- p + \Xi^0 n]$ hyperfragment with $T=0$ there are no selection rules resulting from isotopic spin invariance as for $T=1$. Here the selection rules can arise only from the identity of the particles produced. If Λ has the spin $\frac{1}{2}$ then from the condition that the function of the two Λ particles be antisymmetric, it follows that $L+S$ is even, where L is the relative orbital moment and S is the total spin. If we take the total angular momentum $J=1$, then we obtain $L=S=1$ and the system has the parity -1 . The state $P_{2\Lambda} = +1$ for $J=1$ does not exist. For $J=0$ we have two possibilities, $L=S=0$,

⁽¹⁾ P. ZIELIŃSKI: to be published.

$P_{2\Lambda} = +1$ and $L = S = 1$, $P_{2\Lambda} = -1$. Here there are no restrictions on the parity of the system; the same is true for $J > 1$. In Table I we give the parity for the ΞN system with respect to the parity of the nucleon for $J=1$ under the assumption that the Ξ has spin $\frac{1}{2}$.

TABLE I.

S	L	$P_{\Xi} = 1$	$P_{\Xi} = -1$
1	0	+	-
0	1	-	+
1	1	-	+
1	2	+	-

If we suppose that the ground state is the state with $L=0$, $J=1$ (there may appear a mixture with $L=2$) and that $P_{\Xi} = 1$, then the rapid decay into 2Λ is forbidden and the hyperfragment with $T=0$ decaying via $2\Lambda+\gamma$ can exist.

We now present some conclusions which can be drawn as regards the parity of the Ξ according to which of the hyperfragments discussed above exist. Let us consider several possibilities:

1) The charged hyperfragments Ξ^-n , Ξ^0p do not exist; at least one decay of a neutral hyperfragment via $2\Lambda+\gamma$ is observed. (The non-existence of charged hyperfragments is necessary to exclude the possibility that the observed decay into $2\Lambda+\gamma$ took place from the state $T=1$, $T_3=0$ which, if Ξ^-n and Ξ^0p did not exist also does not exist owing to charge symmetry.) We can establish with large probability, that the Ξ is bound in the state $s=1$ and has the parity $P_{\Xi} = 1$.

2) A neutral hyperfragment decaying into $2\Lambda+\gamma$ does not exist. Then either the ΞN forces are too weak for a bound state to exist or if the system

existed, then there are two possibilities (*):

a) the ground state $S=0$ with any parity of the Ξ ;

b) ground state $S=1$, $P_{\Xi}=-1$.

3) There exist charged hyperfragments; decay into $2\Lambda+\gamma$ are observed. If the frequency of decays into $2\Lambda+\gamma$ is greater than the frequency of occurrence of charged hyperfragments, then some part of the decay via $2\Lambda+\gamma$ must come from the state $T=0$ and we can draw conclusions as in 1). In the case of the same frequency or of small statistics no conclusions can be drawn.

Experimental confirmation of whether any of the above mentioned cases occur is difficult, especially in cases 2) and 3) which require good statistics for the decays into neutral particles. On the other hand, as regards case 1) it is not excluded that with the same accuracy which today allows us to state the non-existence of the Λn and Λp hyperfragments we shall be able to establish the non-existence of the Ξ^-n and Ξ^0p hyperfragments. Then the observation of one decay case into $2\Lambda+\gamma$, or the less probable, but easier to discover process or $2\Lambda+e^++e^-$ would give us very valuable information on the Ξ .

It is tempting to make an analogy between the ΞN hyperfragment and the two nucleon system. The strong interaction schemes suggested at the present

(*) It seems that even if a fast reaction into 2Λ occurs, it is meaningful to speak of a bound state. This results from the fact that for the reactions in light nuclei or for surface interactions in heavy nuclei the ratio of the decay time to the time necessary to create the hyperfragment is

$$\left| \frac{\langle V_{N\Xi}(\pi) \rangle}{\langle V_{K\Xi}(K) \rangle} \right|^2 \frac{\varrho}{\varrho_{2\Lambda}}$$

This expression, for sufficiently large E_{Ξ} is distinctly larger than unity and the existence of the bound state can be established by investigation of the energy of the Λ -particle.

time (²⁻⁴) assume identical coupling through a pion for the Ξ and N, and therefore we can expect that the potentials $V_{NN}(\pi)$ and $V_{N\Xi}(\pi)$ will be qualitatively similar. The Ξ particle and the nucleon are coupled differently with the K field, but also in this case the ΞN hyperfragment such as the deuteron can occur only by the exchange of two K mesons, which corresponds to a potential with a range $1/2M_K$. In addition, taking into account the fact that g_K is probably less than g_π we see that the difference between $V_{NN}(K)$ and $V_{N\Xi}(K)$

should not produce any essential changes. In such a case the ground state of ΞN would be the state $T=0$, $S=1$. The states with $T=1$ would not exist as bound ones and the establishment of the existence or non-existence of ΞN as a bound system would uniquely determine the parity of the Ξ particle as $P_\Xi = 1$ or $P_\Xi = -1$. This analogy cannot be applied to the $2n\Xi^-$ and $2p\Xi^0$ systems because of the Pauli exclusion principle for nucleons, which forces the third nucleon, in $T=\frac{1}{2}$, states to occupy a state with $l=1$.

* * *

(²) M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).
 (³) B. D'ESPAGNAT, J. PRENTKI and A. SALAM: *Nucl. Phys.*, **5**, 447 (1957).
 (⁴) W. KRÓLIKOWSKI: *Nucl. Phys.*, **8**, 461 1957.

In conclusion I would like to thank Prof. W. KRÓLIKOWSKI, Prof. J. WERLE, and Dr. P. ZIELIŃSKI for valuable discussions.

On the Energy Equation for a Gravitating Test Particle.

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(ricevuto il 31 Gennaio 1959)

Introduction.

In a previous paper ⁽¹⁾ we have shown that the equations of motion of a test particle M in a gravitational field, with respect to a general physical system of reference S , can be given a newtonian form

$$(1) \quad \frac{\tilde{d}^* p_\alpha}{dT} = mG_\alpha .$$

In order to prove this it is sufficient to define suitable standard quantities, relative to the system S , and to adopt a particular differential operation \tilde{d}^* (covariant transverse differentiation).

In addition to (1) another scalar equation holds for M , i.e. (I, (90)). This equation, which can be interpreted as the energy equation, will now be examined briefly.

Let us shortly recall some of the results of I; for all the details the reader is referred to the previous paper.

1. - Summary of the previous results.

In the space-time manifold V_4 , let x^i ⁽²⁾ be physically admissible co-ordinates ($x^4 = \text{var.}$ being time-like lines), $ds^2 = g_{ik} dx^i dx^k$ the fundamental quadratic form (hyperbolic of the type $+++-$), and γ a unitary 4-vector tangent to the lines x^4 ($\gamma^i = (0, 0, 0, 1/\sqrt{-g_{44}})$; $\gamma_i = g_{ia}/\sqrt{-g_{44}}$). In every point x of V_4 let Σ_x be the flat 3-space tangent to V_4 and orthogonal to γ . The covariant transverse derivation $\tilde{\nabla}_\alpha^*$ — introduced in the paper I — operates on a vector field v everywhere orthogonal

⁽¹⁾ C. CATTANEO: *Nuovo Cimento*, **10**, 318 (1958). This paper will be in the following quoted as I.

⁽²⁾ Latin indexes vary from 1 to 4; greek indexes vary from 1 to 3.

to γ ($v_4 = 0$) (or on a tensor field everywhere belonging to Σ_x) and produces a tensor $\tilde{\nabla}_\alpha^* v_\beta$ still belonging to Σ_x . Formally such a derivative can be calculated as a usual covariant derivative in a riemannian 3-dimensional continuum with a sort of Christoffel symbols constructed by means of the tensor $\gamma_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha \gamma_\beta$ instead of $g_{\alpha\beta}$, and the systematic substitution of the ordinary partial derivation ∂_α by the transverse partial derivation $\tilde{\partial}_\alpha = \partial_\alpha - (\gamma_\alpha/v_4)\partial_4$ (see: I, (55), (56), (57)). The definition of the absolute transverse differentiation $\tilde{d}^* v_\beta$ is then immediate (I, (53), (63), (64)).

For a moving test particle we consider the absolute quantities m_0 (proper mass), $\mathbf{U} = dx^i/d\tau$ (4-velocity), $\mathbf{P} \equiv m_0 \mathbf{U}$ (4-momentum); and the following relative standard quantities: a) standard spatial metric tensor: $\gamma_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha \gamma_\beta$ ⁽³⁾; b) relative standard time: $dT = -(1/c)\gamma_i dx^i$; c) relative standard 3-velocity: $v^\alpha = dx^\alpha/dT$, $v^2 = \gamma_{\alpha\beta} v^\alpha v^\beta$; d) relative standard mass: $m = m_0(dT/d\tau) = m_0/\sqrt{1 - v^2/c^2}$; e) relative standard 3-momentum: $p^\alpha = P^\alpha = mv^\alpha$. All these quantities, as well as the notion of transverse derivation, are invariant in all internal changes of co-ordinates (I, (4)).

By means of these definitions and the differential operations recalled above, it has been possible to show that if we project orthogonally on Σ_x the absolute equation of motion of a freely falling particle, $D\mathbf{P}/dT = 0$ ⁽⁴⁾, we obtain the three equations (1), where the gravitational field G_α has the form (I, (101))

$$(2) \quad \begin{cases} G_\alpha = G'_\alpha + G''_\alpha, \\ G'_\alpha = -c^2 \left[\frac{1}{2} \tilde{\partial}_\alpha \log(-g_{44}) - \partial_4 \left(\frac{\gamma_\alpha}{\gamma_4} \right) \right], \quad G''_\alpha = c \tilde{\Omega}_{\alpha\beta} v^\beta. \end{cases}$$

The antisymmetric spatial tensor⁽⁵⁾ $\Omega_{\alpha\beta} = \gamma_4 [\tilde{\partial}_\alpha(\gamma_\beta/\gamma_4) - \tilde{\partial}_\beta(\gamma_\alpha/\gamma_4)]$ is the projection on Σ_x of the tensor 4-curl $\Omega_{ij} = \partial_i \gamma_j - \partial_j \gamma_i$. We add here that the last term in (2), G''_α , is invariant with respect to every internal change of co-ordinates (I, (4)) whereas the two former terms are separately invariant only for a spatial change of co-ordinates (I, (5)). However their sum, G'_α , is invariant for a general internal change, as we can recognize, for instance, by putting it in the form $G'_\alpha = c^2 \Omega_{\alpha i} v^i$.

2. - Energy equation.

Let us now consider the fourth equation, (I, (90)), which completes the absolute equation (I, (85)):

$$(3) \quad \frac{DP_4}{dT} \equiv \frac{dP_4}{dT} - \frac{1}{2} \partial_4 g_{ik} P^i \frac{dx^k}{dT} = 0.$$

If in its left-hand side we separate the terms which have one or two indexes equal to 4, and we remember the expression $P^4 = m(dx^4/dT) = -mc(1/\gamma_4 + \gamma_\beta v^\beta/c\gamma_4)$

⁽³⁾ It is the same tensor employed in the transverse covariant derivation. This tensor had already been considered in general relativity; see, for instance: A. LICHNEROWICZ: *Théories relativistes de la gravitation et de l'électromagnétisme* (Paris, 1955), p. 9; C. MÖLLER: *The theory of relativity* (Oxford, 1952), p. 238.

⁽⁴⁾ D is the symbol of absolute differentiation in V_4 .

⁽⁵⁾ We have changed the previous notation writing $\tilde{\Omega}_{\alpha\beta}$ instead of $\gamma_4 \Omega_{\alpha\beta}$ as appears in [I, (101)].

(see: I, (72)), we can give the equation (3) the following form

$$(4) \quad \frac{dP_4}{dT} = m \left[\frac{1}{2} \partial_4 \gamma_{\alpha\beta} v^\alpha v^\beta + e \gamma_4 \partial_4 \left(\frac{\gamma_\alpha}{\gamma_4} \right) v^\alpha - e^2 \frac{\partial_4 \gamma_4}{\gamma_4} \right].$$

In order to give to (4) a physical interpretation, we shall add the following definition: *standard effective relative energy*

$$(5) \quad E = -e \gamma^i P_i = e \frac{P_4}{\gamma_4} = mc^2.$$

As well as the other standard quantities (dT , m , v^α , p^α), E has an invariant meaning with respect to any internal change of co-ordinates and has the same form as the corresponding quantity in the Special Theory ⁽⁶⁾.

If now we substitute to P_4 its equivalent expression $P_4 = mc\gamma_4 = (1/c)\gamma_4 E$, (4) can be written

$$(6) \quad \frac{dE}{dT} = -mc^2 \left[\frac{\tilde{\partial}_\alpha \gamma_4}{\gamma_4} - \partial_4 \left(\frac{\gamma_\alpha}{\gamma_4} \right) \right] v^\alpha + \frac{1}{2} mc^2 \frac{\partial_4 \gamma_{\alpha\beta}}{\gamma_4} v^\alpha v^\beta.$$

The first two terms in the right-hand side are simply the instantaneous power, $dL/dT = mG'_\alpha v^\alpha = mG_\alpha v^\alpha$, of the gravitational force acting on the particle. The last term may be interpreted as the rate of change of the energy of the particle due to the dependence on time of the system of reference.

If the physical fluid of reference is stationary (g_{ik} independent of x^k) the last term vanishes and (6) assumes the classical form

$$(7) \quad dE = dL :$$

the increase of the energy of the particle equals the amount of the work done by the force.

In the same case of a stationary system of reference one can also introduce the definition of the *total relative energy* of the particle, recalled in (I, (84)) and already employed in general relativity (see, for instance, MÖLLER: *op. cit.*, p. 294):

$$(8) \quad H \equiv -eP_4 \equiv -mc^2\gamma_4 \equiv -E\gamma_4.$$

H is not invariant with respect to a general internal change of co-ordinates (it has an invariant character only by respect to a spatial change of co-ordinates [I, (5)]). However it can be noted that also the stationary character of the system of reference is not conserved in a general internal change. So the physical meaning of H is linked to the fluid of reference with a particular system of clocks which makes it stationary.

In such a system (7), or more immediately (4), gives the well known integral of motion (see MÖLLER: *op. cit.*, p. 294):

$$(9) \quad H = \text{const},$$

which justifies for H the name of total energy.

(6) By means of the standard definitions, the gravitational field does not enter explicitly in the definition of the dynamical quantities: it has only a direct influence on the space and time measurements. Occasionally we remember that the adoption of the standard spatial tensor and the standard relative time gives to ds^2 the form [I, (68)]: any optic anisotropy of the space disappears, the speed of the light being equal to c in every direction.

On the $3d$ Wave Functions in Atoms of the Iron Group.

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(ricevuto il 4 Febbraio 1959)

Among the parameters which appear in the theory of the hyperfine structure and quadrupole interaction of the iron transition group (¹) there is the average value of r^{-3} for the $3d$ shell. Since the experimental values of $\langle r^{-3} \rangle$ for a certain number of elements have been established (²) it seems interesting to try to calculate theoretically this parameter by choosing the appropriate wave functions for the elements with atomic number, Z , from 21 to 29. A theoretical calculation of $\langle r^{-3} \rangle$ for Cu has been performed using Hartree self-consistent wave functions (³). Numerical tables for all $3d$ wave functions are not available and in every case require numerical integration. For this reason the use of simple analytical forms is advisable.

The most simple analytical form of wave functions has been given by SLATER (⁴) and analytical expressions less

simple than those of Slater have been suggested by LÖWDIN and APPEL (⁵) (Table V of ref. (⁵)). However, the functions of Slater and of Löwdin and Appel cannot be used in the calculation of $\langle r^{-3} \rangle$ since the main contribution comes from the part of the wave function within 0.5 atomic units (this region is smaller for larger Z ; for example, for Copper it is ~ 0.3 a.u.) of the nucleus and in this region these functions are inaccurate. What we need, therefore, is a wave function accurate within this distance of the nucleus.

In the paper of ref. (⁵) LÖWDIN and APPEL suggest simple expressions for $3d$ wave functions in a region very close to the nucleus. These functions are given by the following formula:

$$f_{3d}(Z, r) = C_3(Z) r^3 \exp\left(-\frac{Z}{3}r\right),$$

with the coefficients $C_3(Z)$ given in Table VI of ref. (⁵). These wave functions are of a conveniently simple form

(¹) A. ABRAGAM and M. H. L. PRYCE: *Proc. Roy. Soc.*, **205**, 135 (1951).

(²) A. ABRAGAM, J. HOROWITZ and M. H. L. PRYCE: *Proc. Roy. Soc.*, **230**, 169 (1955).

(³) R. M. STERNHEIMER: *Phys. Rev.*, **405**, 158 (1957).

(⁴) J. C. SLATER: *Phys. Rev.*, **38**, 57 (1930).

(⁵) P. LÖWDIN and K. APPEL: *Phys. Rev.*, **403**, 1746 (1956).

but agree with numerical tables deduced by the self-consistent method for V²⁺ (6), Mn²⁺ and Cu⁺ (7) only for very small values of r . Nevertheless it is always possible to use an equally simple form with the same coefficients C_3 by modifying properly the exponential, e.g. an expression of the type:

$$(1) \quad f_{3d}(Z, r) = C_3 r^3 \exp\left(-\frac{Z}{x}r\right).$$

It would perhaps be more appropriate to introduce a Z_{eff} but for ease of calculation it is simpler to vary the denominator. Using the numerical self-consistent wave function for V²⁺, Mn²⁺, Cu⁺ we attempt to fix x in such a way that (1) is in accordance with these wave functions. When this is done for the three preceding elements we find that the best values for x satisfy the following linear equation:

$$(2) \quad x = 7.50125 - 0.12575Z.$$

We assume that (2) is valid for all the elements in the iron transition group. The value of C_3 for Cu has been taken by extrapolation and is approximately equal to 251.5.

The value of $\langle r^{-3} \rangle$ has been calculated using (1) and (2) in the usual manner:

$$\langle r^{-3} \rangle = \int_0^\infty (R^2(r)/r^3) r^2 dr,$$

in which $R(r)$ is the normalized radial part of the 3d wave function. Since the part of $rR(r)$ outside 0.5 does not contribute to the integral and since $[rR(r)]$, near the origin, is well approximated by f_{3d} we have

$$(3) \quad \langle r^{-3} \rangle \cong \int_0^\infty \frac{f_{3d}^2(r)}{r^3} dr = \frac{6C_3^2}{(2z)^4} x^4.$$

(6) B. H. WORSLEY: *Proc. Roy. Soc.*, **247**, 390 (1958).

(7) D. R. HARTREE: *The Calculation of Atomic Structures* (New York, 1956).

The numerical values obtained using (3) are in very good agreement with the experimental values (2) as is shown in Table I.

TABLE I. — *Experimental and calculated values of $\langle r^{-3} \rangle$ (a.u.).*

Element	x	$\langle r^{-3} \rangle$ exp.	$\langle r^{-3} \rangle$ calc.
Sc	4.86050	1.2	1.20
Ti	4.73475	—	1.70
V	4.60900	2.3	2.30
Cr	4.48325	—	2.98
Mn	4.35750	3.4 ₅	3.74
Fe	4.23175	—	4.54
Co	4.10600	5.2	5.37
Ni	3.98025	—	6.20
Cu	3.85450	7.4 ₅	7.40

An interesting question is raised by the remark that the observed hfs from which the experimental values have been obtained has been interpreted in terms of an isotropic part from unpaired s-electrons and an anisotropic part from d-electrons. The explanation of the isotropic part is not a one-electron process and one may ask whether the anisotropic part is a one-electron process. The agreement between the experimental values of $\langle r^{-3} \rangle$ and those calculated by using simple analytical one-electron wave functions leads us to presume that one can understand the hfs in terms of reasonable wave functions, and that, for example, one does not have shielding and anti-shielding effects such as are obtained in nuclear quadrupole spectroscopy.

* * *

The author would like to thank Prof. STEVENS, Prof. FUMI and Prof. PALUMBO for useful discussions. He wishes to thank Prof. SANTANGELO for his continuing interest during the course of this work.

A Possible Experimental Test of the Relative Parity of Charged and Neutral K's.

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(ricevuto il 6 Febbraio 1959)

In some recent attempts by GELL-MANN⁽¹⁾, SCHWINGER⁽²⁾ and particularly by PAIS⁽³⁾ the idea has been developed of examining the consequences of introducing in the Lagrangian of the strong interactions rather high symmetries. In particular PAIS⁽³⁾ has noted that a too high degree of symmetry is incompatible with the experimental data, because it implies some incorrect relations between the observed rates of some processes and also the vanishing of certain experimentally non zero cross sections.

To avoid such unpleasant consequences of this excess of symmetry one has to destroy part of it. PAIS⁽⁴⁾ investigates whether it is possible to do this by assuming opposite parity for the charged and neutral K and introducing an appropriate KK π interaction (the strong interactions are still assumed to conserve parity).

Notice that if the charged and neutral K have opposite parity the [K] interactions destroy the charge symmetry.

The purpose of the present note is simply that of pointing out an experiment which is practically feasible and which should be appropriate to decide: *a*) whether charge symmetry is violated in the [K] interactions; *b*) whether such violation, if it exists, has to be attributed to a parity difference.

We start with some remarks: obviously the first possibility to which one may think for examining a parity difference between the charged and neutral K consists, as Pais has noted, in exploring the reaction $K + \text{Nucleon} \rightarrow K + \text{Nucleon}$, with and without charge exchange. A different low energy behaviour of the charge exchange versus the non charge exchange cross section should be indicative of opposite parities of charged and neutral K. However for incident K^- there are in such reactions, too many channels open; though events both of the kind $K^- + p \rightarrow K^- + p$ and of the kind $K^- + p \rightarrow K^0 + n$, have been observed, a quantitative comparison of the energy dependence of the above reactions seems in practice to be dif-

(¹) M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

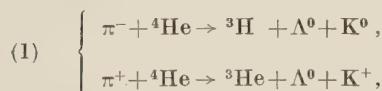
(²) J. SCHWINGER: *Ann. Phys.*, **2**, 407 (1957).

(³) A. PAIS: *Phys. Rev.*, **110**, 574 (1958).

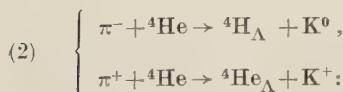
(⁴) A. PAIS: *Phys. Rev.*, **112**, 625 (1958); we shall use the notation of references (³) and (⁴).

ficult. The situation would be much better if it might be possible to perform the charge exchange and non exchange reactions of K^+ on free neutrons; there infact only the two channels $K^+ + n \rightarrow K^+ + n$ and $K^+ + n \rightarrow K^0 + p$ are open; however the most free neutrons of which we can dispose are to be found in the deuteron and there the effect of the Pauli principle introduces a spurious inhibition of the exchange reaction which has nothing to do with our problem.

The situation is, on the contrary, experimentally much more favourable and clean if we consider the reactions induced by pions. The obvious method for investigating the charge symmetry is that of bombarding a self conjugate nucleus with π^+ and π^- and compare the rates of the «mirror»⁽⁵⁾ processes for equal energy of the π^+ or π^- ; in our case the most favourable nuclei are probably D or ${}^4\text{He}$. In this way it should be possible, very simply, to give an answer to question a). As to question b) the answer can be found simply in comparing the angular distributions and energy dependence of the various «mirror» processes induced by the π^+ and π^- ; a difference in parity would certainly imply strong differences in these angular distributions and energy dependence near threshold. In particular, considering the reactions in ${}^4\text{He}$ the most appropriate and probably most frequent reactions of associated production induced by π^- and respectively π^+ below the Σ threshold should be:



and also:



⁽⁵⁾ The word «mirror» refers here to the simultaneous change of π^+ in π^- , p in n , K^+ in K^0 , Σ^+ in Σ^- .

Both the reactions (1) and (2) should be very clearly interpreted in a ${}^4\text{He}$ bubble chamber.

We conclude adding a few points:

1) It is clearly not very «aesthetic» to gain the high symmetry of the doublet approximation, ⁽⁶⁾ loosing the old good charge symmetry; so we cannot help from expressing our hope that the reactions considered above really behave as mirror reactions.

2) One way to conserve the advantages of the doublet approximation without renouncing to charge symmetry might consist in renouncing to parity conservation in some «moderately» strong interactions; writing the doublet approximation as it is written for even p(K), but adding, *for instance*, the $KK\pi$ interaction.

3) If it should however happen that in the reaction $\Sigma + N \rightarrow \Lambda + N$ parity is *too strongly* ⁽⁷⁾ violated, one would probably have to renounce also to the doublet approximation (at least without derivative couplings); infact if we insist on having charge independence and time reversal for the $[\Sigma\Sigma\pi]$ interaction, such interaction *must* conserve parity, and the same should be true, in the spirit of the doublet approximation, for the $[\Sigma\Lambda\pi]$ interaction.

Note added in proof.

The same experiment has been proposed by A. PAIS in the December 1, 1958 issue of *Phys. Rev. Letters*, which was available here only after this paper had been submitted for publication.

⁽⁶⁾ Compare, in addition to ref. ⁽⁸⁾, J. TIOMMO; *Nuovo Cimento*, **6**, 69, (1957); N. DALLAPORTA; *International Conference on Mesons and Recently Discovered Particles*, V-3 and *Nuovo Cimento*, **7**, 200 (1958).

⁽⁷⁾ This means: if it should be necessary to attribute such violation to the strong pion-baryon interaction.

Note on Møller's Energy-Momentum Pseudo-Tensor (*).

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In a recent paper (¹) C. MØLLER has shown that the energy-momentum pseudo-tensor of matter and gravitation introduced by him (²) may be obtained by varying the curvature-scalar density in contrast to the Einstein energy-momentum pseudo-tensor that is obtained by varying the pseudo-scalar density (³)

$$(1) \quad \mathfrak{L} = -g^{\mu\nu} (\Gamma^\beta_{\alpha\mu} \Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\alpha\beta} \Gamma^\alpha_{\mu\nu}).$$

Møller's pseudo-tensor has the advantage over Einstein's pseudo-tensor that it gives an energy-density and energy-current transforming as a scalar- and a vector-density respectively when purely spatial transformations are performed.

In (M2) the variation in question is performed with the curvature-scalar density \mathfrak{R} regarded as a function of $g^{\mu\nu}$ and their first and second order derivatives. It is shown in this note that a simpler procedure is to regard

$$(2) \quad \mathfrak{R} = g^{\mu\nu} R_{\mu\nu},$$

as a function of the variables $g^{\mu\nu}$, $\Gamma^\alpha_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma,\delta}$ (⁴). $R_{\mu\nu}$ being the ordinary composition of $\Gamma^\alpha_{\beta\gamma}$ and their derivatives.

The local variations in $g^{\mu\nu}$ and $\Gamma^\alpha_{\beta\gamma}$ induced by the coordinate transformation

$$(3) \quad x'^\mu = x^\mu + \xi^\mu(x),$$

are in our case

$$(4) \quad \delta X^A = u^A{}_\beta \xi^\beta_{,\alpha} + v^A{}_{\beta\gamma} \xi^\beta_{,\alpha\gamma} - Y^A{}_{,\alpha} \xi^\alpha,$$

(*) This investigation was sponsored by the U. S. Air Force.

(¹) C. MØLLER: *The energy-momentum complex in the general theory of relativity*. To be published in *Dan. Mat. Fys. Medd.* In the following denoted (M2).

(²) C. MØLLER: *Ann. Phys.*, **4**, 347 (1958). In the following denoted (M1).

(³) $g^{\mu\nu} = \sqrt{-g} \cdot g^{\mu\nu}$.

(⁴) I owe many thanks to prof. O. KLEIN for proposing the use of this method, the Palatini method, in the present case.

The Y^A can stand for the $g^{\mu\nu}$ as well as for the $T^\lambda_{\mu\nu}$. In the former case

$$(5) \quad u^{\mu\nu\alpha}_{\beta} = g^{\alpha\mu} \delta_{\beta}^{\nu} + g^{\alpha\nu} \delta_{\beta}^{\mu} - g^{\mu\nu} \delta_{\beta}^{\alpha},$$

$$(6) \quad v^{\mu\nu\alpha\gamma}_{\beta} = 0,$$

while in the latter case

$$(7) \quad u^{\lambda}_{\mu\nu\beta} = \Gamma^{\lambda}_{\mu\nu} \delta_{\beta}^{\lambda} - \Gamma^{\lambda}_{\beta\nu} \delta_{\mu}^{\lambda} - \Gamma^{\lambda}_{\beta\mu} \delta_{\nu}^{\lambda},$$

$$(8) \quad v^{\lambda}_{\mu\nu\beta} = -\frac{1}{2} (\delta_{\beta}^{\lambda} \delta_{\mu}^{\nu} \delta_{\nu}^{\lambda} + \delta_{\beta}^{\lambda} \delta_{\nu}^{\lambda} \delta_{\mu}^{\nu}).$$

The variation of \Re is very easily obtained. Using $g^{\mu\nu}{}_{;\kappa} = 0$ one finds that (5)

$$(9) \quad \delta\Re_{\mu\nu} = R_{\mu\nu} \delta g^{\mu\nu} - (g^{\mu\nu} \delta \Gamma^{\kappa}_{\mu\nu})_{;\kappa} + (g^{\mu\nu} \delta \Gamma^{\kappa}_{\mu\kappa})_{;\nu}.$$

From (9) and (6) it follows that

$$(10) \quad \frac{\delta\Re}{\delta Y^A} v^A{}^{\alpha\gamma}_{\beta} = 0.$$

Using this formula, equation (30) in (M2):

$$(11) \quad \frac{\delta\Re}{\delta Y^A} Y^A{}_{;\alpha} + \left(\frac{\delta\Re}{\delta Y^A} u^A{}^{\beta}_{\alpha} \right)_{;\beta} = 0,$$

is shown to be true also in our case.

The fact that \Re is a scalar density yields the following local variation

$$(12) \quad \delta\Re = -(\Re \xi^{\alpha})_{,\alpha}.$$

Using (9) and (4) in the left hand side of this and putting all the coefficients of the different derivatives of ξ^{α} equal to zero gives the formulae corresponding to (33)–(39) in (M2):

$$(13) \quad S_{\alpha}{}^{\beta}{}_{,\beta} = 0,$$

$$(14) \quad S_{\alpha}{}^{\beta} = V_{\alpha}{}^{\beta\gamma}{}_{,\gamma},$$

$$(15) \quad V_{\alpha}{}^{\beta\gamma} + V_{\alpha}{}^{\gamma\beta} + 2V_{\alpha}{}^{\beta\gamma\delta}{}_{,\delta} = 0,$$

$$(16) \quad V_{\alpha}{}^{\beta\gamma\delta} + V_{\alpha}{}^{\delta\beta\gamma} + V_{\alpha}{}^{\gamma\delta\beta} = 0; \quad V_{\alpha}{}^{\beta\gamma\delta} = V_{\alpha}{}^{\gamma\beta\delta},$$

where

$$(17) \quad S_{\alpha}{}^{\beta} = -\frac{\delta\Re}{\delta Y^A} u^A{}^{\beta}_{\alpha} + \frac{\delta\Re}{\delta Y^A} Y^A{}_{,\alpha} - \delta_{\alpha}^{\beta} \Re,$$

$$(18) \quad V_{\alpha}{}^{\beta\gamma} = \frac{\delta\Re}{\delta Y^A} u^A{}^{\beta}_{\alpha},$$

$$(19) \quad V_{\alpha}{}^{\beta\gamma\delta} = \frac{\delta\Re}{\delta Y^A} v^A{}^{\beta\gamma}_{\alpha}.$$

⁽⁵⁾ See f.i. C. MØLLER: *The Theory of Relativity* (Oxford, 1952), p. 334.

Notice that our $V_{\alpha}^{\beta\gamma\delta}$ have not exactly the same symmetry properties as the corresponding quantities in (M2), but still an antisymmetric quantity $U_{\alpha}^{\beta\gamma}$ fulfilling

$$(20) \quad S_{\alpha}^{\beta} = U_{\alpha}^{\beta\gamma,\gamma},$$

$$(21) \quad U_{\alpha}^{\beta\gamma} + U_{\alpha}^{\gamma\beta} = 0,$$

can be constructed. It is easy to see from (14)–(16) that

$$(22) \quad U_{\alpha}^{\beta\gamma} = \frac{1}{2} V_{\alpha}^{\beta\gamma} + \frac{1}{3} V_{\alpha}^{\gamma\delta\beta,\delta} - \underbrace{(\beta,\gamma)},$$

fulfils (20) and (21). This formula corresponds to (46) in (M2).

It is now a simple matter to calculate S_{α}^{β} and $U_{\alpha}^{\beta\gamma}$ explicitly from (17), (22), (9) and (5)–(8).

We obtain

$$(23) \quad S_{\alpha}^{\beta} = -2R_{\alpha}^{\beta} - g^{\mu\nu}(I_{\mu\nu}^{\beta} - \delta_{\nu}^{\beta} I_{\mu\nu}^{\infty})_{,\alpha}.$$

Using the formulae ⁽⁶⁾

$$(24) \quad \frac{\partial \mathcal{L}}{\partial g^{\mu\nu},\alpha} = I_{\mu\nu}^{\alpha} - \frac{1}{2} \delta_{\mu}^{\alpha} I_{\nu\nu}^{\infty} - \frac{1}{2} \delta_{\nu}^{\alpha} I_{\mu\nu}^{\infty},$$

and

$$(25) \quad \sqrt{-g} \vartheta_{\alpha}^{\beta} = \frac{1}{2\zeta} \left\{ \frac{\partial \mathcal{L}}{\partial g^{\mu\nu},\beta} g^{\mu\nu,\alpha} - \delta_{\alpha}^{\beta} \mathcal{L} \right\},$$

S_{α}^{β} in (23) is easily seen to be identical with $\mathcal{C}_{\alpha}^{\beta}$ in formula (106) of (M1).

Further we obtain after a short and straightforward calculation

$$(26) \quad U_{\alpha}^{\beta\gamma} = g^{\beta\sigma} I_{\gamma}^{\alpha\sigma} - g^{\gamma\sigma} I_{\alpha}^{\beta\sigma}.$$

By expressing the $I_{\alpha}^{\beta\gamma}$ in terms of the $g_{\mu\nu}$ and their first derivatives, eq. (26) gives

$$(27) \quad U_{\alpha}^{\beta\gamma} = \sqrt{-g} g^{\beta\sigma} g^{\gamma\sigma} (g_{\alpha\sigma,\theta} - g_{\alpha\theta,\sigma}).$$

This is precisely the expression for $\chi_{\alpha}^{\beta\gamma}$ given in formula (65) of (M1).

* * *

The author wishes to thank professor O. KLEIN and professor C. MØLLER for the discussion in which this investigation was suggested. I am also indebted to Professor Møller for his hospitality at Nordita, where this note was written.

⁽⁶⁾ See *f.i.* TOLMAN: *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), pp. 222-224.

LIBRI RICEVUTI E RECENSIONI

F. G. FRIEDLANDER - *Sound Pulses*, volume in-16° della Cambridge University Press, pubblicato il 27-6-58; 40 scellini. Fa parte della raccolta « Cambridge Monographs on Mechanics and Applied Mathematics ».

Il volume *Sound Pulses* di FRIEDLANDER tratta la propagazione di perturbazioni non periodiche ed impulsi sonori, fondandosi sulla teoria delle equazioni differenziali lineari a derivate parziali di tipo iperbolico.

Nel primo capitolo si stabilisce la forma generale dell'equazione di propagazione di piccole perturbazioni in un mezzo omogeneo e si dimostra che la soluzione di Poisson dell'equazione gode della fondamentale proprietà di propagazione.

Nel secondo capitolo si estende tale proprietà al caso più generale del mezzo non omogeneo e si discutono i problemi al contorno che sorgono.

Nel terzo capitolo si estendono i risultati precedenti ai fronti d'onda d'urto, dove la pressione è discontinua e si pongono le basi dell'acustica delle onde d'urto, che per la sua analogia con l'ottica geometrica viene battezzata « acustica geometrica ».

I capitoli seguenti, quarto, quinto e sesto, si occupano di problemi di riflessione e rifrazione ed infine di problemi di diffrazione delle perturbazioni non periodiche, sulla base dei risultati fondamentali acquisiti nei primi tre capitoli del volume.

Di notevole interesse per il fisicomateematico, l'opera sarebbe di proficua

lettura per molti specialisti di altre branche e precisamente per chi si interessa di aerodinamica, di acustica teorica o applicata, di sismologia o di prospezione geologica.

Purtroppo però il volume riesce di lettura abbastanza difficile per i non iniziati e la trattazione resta troppo spesso puramente matematica cioè assai lontana dalla mentalità dello sperimentale.

F. S. GAETA

Dislocations and mechanical properties of crystals, edito da J. C. FISHER, W. G. JOHNSTON, R. THOMSON e T. VREELAND Jr. (J. Wiley, New York, 1957), pag. XIV + 634.

Il libro raccoglie i lavori discussi alla Conferenza di Lake Placid, New York, dedicata alle osservazioni sperimentali delle dislocazioni ed alla discussione dell'influenza di queste imperfezioni lineari sulle proprietà meccaniche dei solidi. I lavori presentati appaiono distribuiti in otto sezioni principali, riguardanti le osservazioni dirette di dislocazioni, la deformazione plastica di monocristalli « puri », l'indurimento da lavorazione a freddo e la ricottura, le proprietà meccaniche di leghe metalliche e cristalli impuri, il comportamento di solidi sotto sforzi alternati, argomenti particolari della teoria delle dislocazioni, i « baffi di gatto » e cristalli sottili, e il danneggiamento da radiazione. Di particolare interesse è la sezione dedicata alle osservazioni dirette di dislocazioni, che sono

recentemente venute a comprovare la vasta massa di lavoro teorico compiuto dal 1940 nel campo delle dislocazioni. Prove dirette dell'esistenza di dislocazioni nei solidi, e di varie loro proprietà quali l'interazione con difetti puntiformi e moti di scorrimento sono state ottenute con tecniche varie, quali decorazione con particelle colloidali di precipitato, studio di figure d'attacco chimico e termico, microscopia elettronica, e sono descritte per NaCl da AMELINCKX, per Si da DASH, per gli alogenuri d'argento da MITCHELL, per Al da HIRSCH e collaboratori, per LiF da GILMAN e JOHNSTON, per Ag da MACHLIN. Di notevole interesse sono pure gli studi del comportamento di solidi sotto sforzi alternati, che mirano a dare una spiegazione del fenomeno di affaticamento dei metalli, e che forniranno probabilmente nei prossimi anni valide informazioni sulle interazioni fra dislocazioni e difetti puntiformi mediante tecniche di anelasticità. Degne di menzione appaiono infine le osservazioni sperimentali di fenomeni plastici descritte per Cu da BLEWITT, per KCl da T. SUZUKI, e per leghe metalliche da H. SUZUKI e da HONEYCOMBE, la discussione

sione delle proprietà plastiche dei metalli a struttura compatta ripresa da SEEGER, e le note di LEIBFRIED sulla interazione di dislocazioni con vibrazioni reticolari.

L'opera degli editori è stata particolarmente brillante nel corredare ogni lavoro di un resoconto dettagliato delle discussioni che l'hanno seguito: è così possibile rendersi conto dei punti su cui esiste ancora disaccordo o che non sono ancora ben compresi. L'effettiva vitalità ed attualità degli argomenti trattati, su cui è riportata l'opinione dei maggiori esperti del campo, rendono il libro raccomandabile sia ai ricercatori in fisica dello stato solido che ai metallurghi. Naturalmente, poichè il libro fissa un particolare momento di un campo di ricerca in continuo e attivissimo sviluppo, alcuni punti di vista in esso espressi appaiono già superati a due anni di distanza: ricordiamo a questo proposito il modello proposto da GILMAN e JOHNSTON per la generazione di anelli di dislocazione in cristalli temprati, ed il meccanismo discusso da FRIEDEL per la ricottura di cristalli contenenti posti vacanti in soprassaturazione.

M. P. TOSI

PROPRIETÀ LETTERARIA RISERVATA